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A three tangent congruent circle problem

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Abstract. We generalize a sangaku problem involving three congruent tangent circles.

Keywords. sangaku problem, congruent tangent circles, division by 0

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1. Introduction

In this note we generalize the following problem involving three congruent tangent circles (see Figure 1).

Problem 1.1. *Let $ACDE$ be a square with a point B on the side DE. The inradius of the triangle BCD is r, and one of two mutually touching circles of radius r touches the sides BE and AE, and the other touches the sides AB and AE. Show that the inradius of the triangle ABC equals $2r$.*

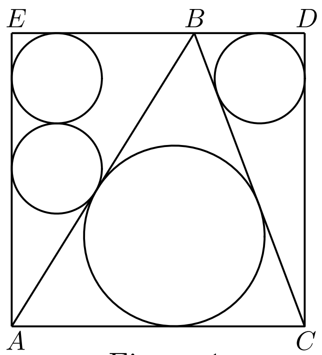

Figure 1.

The problem was proposed by Kobayashi (小林捨吉) and Yagawa (矢川雄七郎) in a sangaku dated 1849 [4], and can also be found in [5]. A similar problem in which $ACDE$ is a rectangle was proposed by Uchida (内田久之丞) [7], [8]. Solutions of those problems can be found in [1, p. 18], [3], [6], [7] and [8]. A generalization in which there are arbitrary number of tangent circles of the same radius in the triangles BCD and BAE can be found in [6]. In this paper we give another generalization of Problem 1.1.

2. Generalization

Let H be the foot of perpendicular from B to CA in Figure 1. The rotation through 180° about the midpoint of BC takes CD to BH and the incircle of BCD to the incircle of BCH, and also the rotation through 180° about the midpoint of AB takes AE to BH and the two circles in BAE to the two circles in BAH (see Figure 2). The problem is generalized as follows (see Figures 3 to 8 and 10 to 14):

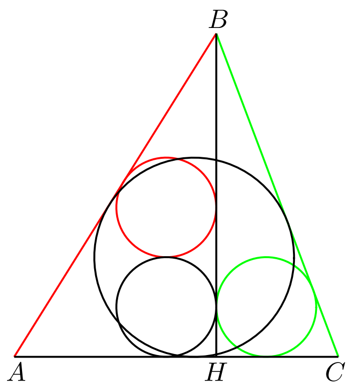

Figure 2.

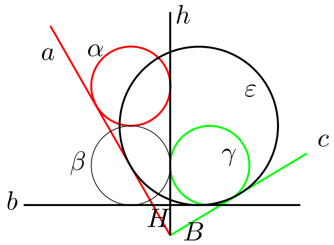

Figure 3: $t < -2r$

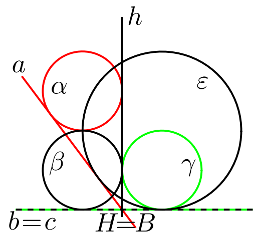

Figure 4: $t = -2r$

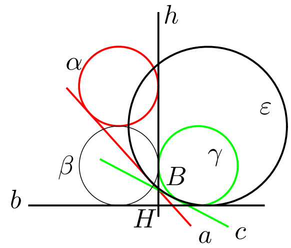

Figure 5: $-2r < t < -\sqrt{2}r$

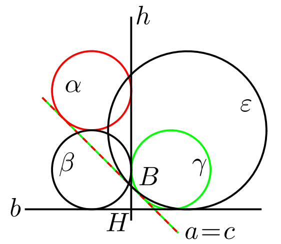

Figure 6: $t = -\sqrt{2}r$

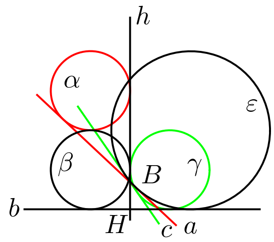

Figure 7: $-\sqrt{2}r < t < -r$

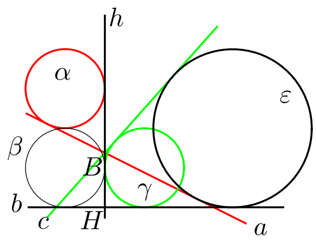

Figure 8: $-r \leq t < 0$

Theorem 2.1. *Let us assume that α, β and γ are circles of radius r such that β touches perpendicular lines b and h meeting in a point H, α is the reflection of β in the remaining tangent of β parallel to b, and γ is the reflection of β in h. Let B be a point on h, and let a (resp. c) be the tangent of α (resp. γ) from B different from h if B is not the point of tangency of α (resp. γ) and h, otherwise*

$a = h$ (resp. $c = h$). If $|BH| \neq 2r$, there is a circle of radius $2r$ touching the line a (resp. b, c) from the same side as α (resp. β, γ).

Proof. We set up a rectangular coordinate system so that the centers of α and γ have coordinates $(-r, r)$ and $(r, -r)$, respectively, i.e., the x-axis overlaps with the remaining external common tangents of β and γ, and the y-axis overlaps with h. Let $(0, t)$ be the coordinates of B. If $t \neq 0$, let ε be the circle of radius $2r$ with center with coordinates $(x_\varepsilon, y_\varepsilon) = (-2r^2/t, 0)$. Let $t_i = t - ir$, $u_a = t_2 t$, $v_a = -2rt_1$, $f_a(x, y) = u_a x + v_a y + 2rt_1 t$ and $s_a = \sqrt{u_a^2 + v_a^2}$, and let $u_c = t_{-2}t$, $v_c = 2rt_{-1}$, $f_c(x, y) = u_c x + v_c y - 2rt_{-1}t$ and $s_c = \sqrt{u_c^2 + v_c^2}$. Then $f_a(0, t) = 0$, $s_a = t_1^2 + r^2$, $f_a(-r, r)/s_a = r$ and $f_a(x_\varepsilon, y_\varepsilon)/s_a = 2r$. Hence $f_a = 0$ is an equation of a, and α and ε touch a from the same side. Also $f_c(0, t) = 0$, $s_c = t_{-1}^2 + r^2$, $f_c(r, -r)/s_c = -r$ and $f_c(x_\varepsilon, y_\varepsilon)/s_c = -2r$. Therefore $f_c = 0$ is an equation of c, and γ and ε touch c from the same side. The rest of the theorem is obvious. □

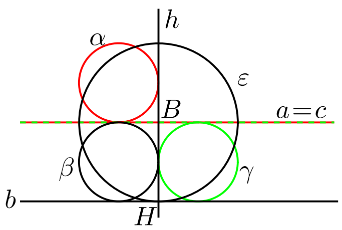

Figure 9: $t = 0$

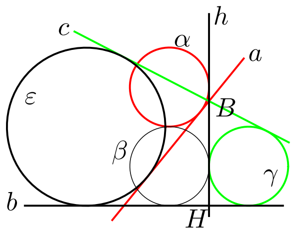

Figure 10: $0 < t < r$

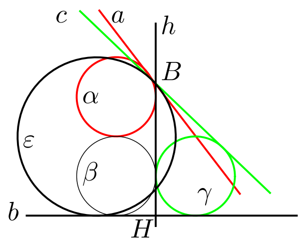

Figure 11: $r \leq t < \sqrt{2}r$

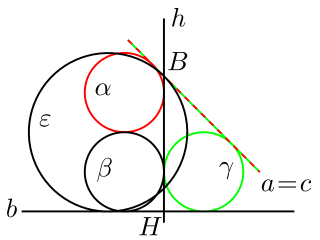

Figure 12: $t = \sqrt{2}r$

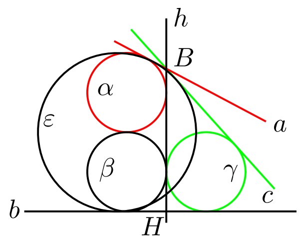

Figure 13: $\sqrt{2}r < t < 2r$

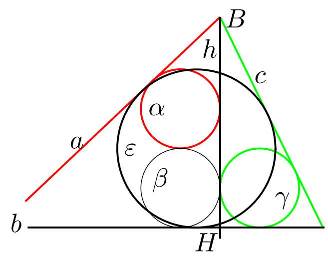

Figure 14: $2r \leq t$

The case $2r < t$ was considered in [1, 3, 4, 5, 6, 7, 8]. The point of tangency of a (resp. c) and ε moves on ε counterclockwise (resp. clockwise) when the value of t increases (see Figures 3 to 8 and Figures 10 to 14 in these orders). If $t = -2r$, then c coincides with b, and γ, ε and c touch at the point with coordinates $(r, -2r)$ (see Figure 4). Let $m_a = -u_a/v_a$ and $m_c = -u_c/v_c$ in the case $v_a v_c \neq 0$. Solving the equation $m_a = m_c$ for t, we get that the lines a and c coincide if and only if $t = \pm\sqrt{2}r$ or $t = 0$. If $t = \pm\sqrt{2}r$, then $m_a = -1$ and ε touches a at B (see Figures 6 and 12). Let us assume $t = 0$. Then B coincides with the origin and a and c coincide with the x-axis, i.e., $m_a = 0$ (see Figure 9). While we have $-2r^2/t = 0$ in the sense of the division by zero [2]. Therefore if ε is still the circle of radius $2r$ with center with coordinates $(-2r^2/t, 0)$, the center of ε coincides with the origin. Each of the tangents at the points of intersection of ε and a is parallel to h. Therefore they have slope $\tan 90°$, where also notice that $\tan 90°$ has meaning and equals 0 in the sense of the division by zero. Therefore the slopes of the tangents and a are the same. Hence we can still consider that a and c touch ε in this case.

Solving the equation $m_a m_c = -1$ for t, we get that a and c are perpendicular if and only if $t = \left(\pm 1 \pm \sqrt{3}\right) r$, and $m_a = \pm 3^{\pm \frac{1}{2}}$ in this event. Let us consider the case $t = \left(1 + \sqrt{3}\right) r$. Let A (resp. C) be the point of intersection of a (resp. c) and b (see Figure 15). Since $m_a = 1/\sqrt{3}$, we have $\angle BAC = 30°$. The reflection of a in the line joining C and the center of ε is the perpendicular to CA touching ε, while $2|BC| = |CA|$. Hence the perpendicular bisector of CA touches ε. Let E and α' (resp. D and γ') be the images of H and α (resp. γ) by the rotation through 180° about the midpoint of AB (resp. BC). Figure 16 is made by ABC, BCD, ε, α' and γ' with their reflections in the perpendicular bisector of CA together with several added circles of radius r and line segments.

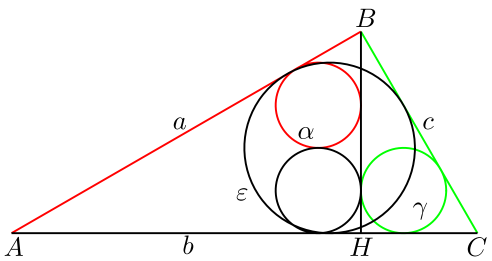

Figure 15.

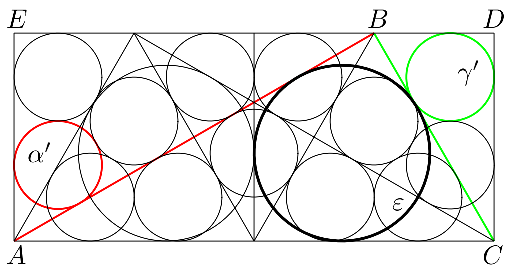

Figure 16.

3. Triangles with height equal to the base

In Figure 2, $|CA| = |BH|$ holds. In this section we characterize triangles with this property in a general way.

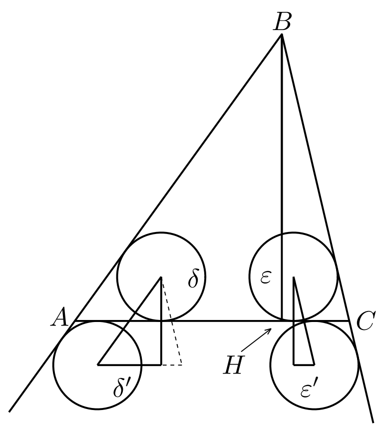

Figure 17.

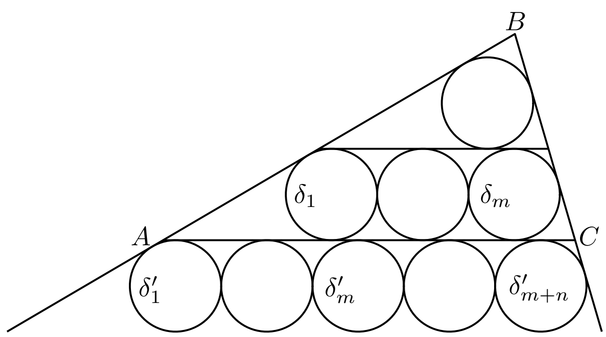

Figure 18: $n = 2$

Proposition 3.1. *Let H be the foot of perpendicular from B to CA for a triangle ABC with inradius r_0. Let δ and ε be the circles of radius $r \leq r_0$ such that δ (resp. ε) touches the sides CA and AB (resp. BC) from the inside of ABC. Let δ' and ε' be the circles of radius r touching the side CA from the side opposite to B such that δ' (resp. ε') touches the line AB (resp. BC) from the same side as δ (resp. ε). If the distance between the centers of δ and ε (resp. δ' and ε') equals d (resp. d'), we have*

$$\frac{d' - d}{2r} = \frac{|CA|}{|BH|}. \tag{1}$$

Proof. If we translate the circles ε and ε' so that the image of ε coincides with δ, then the centers of the circles δ' and δ and the center of the image of ε' form a triangle similar to ABC (see Figure 17). Hence we have (1). □

The proposition shows that $|CA| = n|BH|$ and $d' - d = 2nr$ are equivalent for a natural number n. Hence we have the next theorem (see Figure 18).

Theorem 3.1. *Let H be the foot of perpendicular from B to CA for a triangle ABC. Let δ_1, δ_2, $\cdots$, δ_m be the circles of radius r such that they touch the side CA from the inside of ABC and δ_1 touches the side AB, δ_i $(i = 2, 3, \cdots, m)$ touches δ_{i-1} from the side opposite to A and δ_m touches the side BC. Then $|CA| = n|BH|$ for a natural number n if and only if there exist circles δ'_1, δ'_2, $\cdots$, δ'_{m+n} of radius r such that they touch the side CA from the side opposite to B, δ'_1 touches the line AB from the same side as δ_1, δ'_i $(i = 2, 3, \cdots, m+n)$ touches δ'_{i-1} from the side opposite to A, and δ'_{n+m} touches the line BC from the same side as δ_m.*

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