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A note on the arbelos in Wasan geometry, a problem in Sampō Tenzan Tebikigusa Furoku

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Abstract. We consider a problem in Wasan geometry involving a special arbelos configuration and give a condition that the figure can be constructed, and show the existence of two pairs of two non-Archimedean congruent circles.

Keywords. arbelos, non-Archimedean congruent circles.

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1. Introduction

We consider the arbelos appeared in Wasan geometry. Let us consider an arbelos formed by three semicircles α, β and γ with diameters AO, BO and AB, respectively for a point O on the segment AB (see Figure 1). We call the radical axis of α and β the axis. The points of intersection of γ and the axis is denoted by I. Let a and b be the radii of α and β, respectively. In this note we consider the following problem in [1] (see Figure 2).

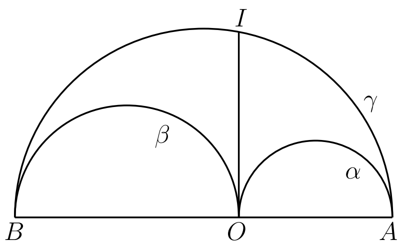

Figure 1.

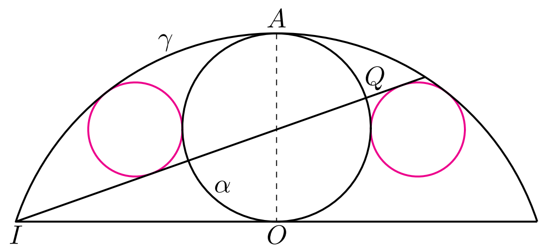

Figure 2.

Problem 1. For a point Q on the reflection of α in the line AO, the incircle of the curvilinear triangle made by α, γ and the line IQ and the circle touching the reflections of α and γ in AO and IQ from the side opposite to the incircle are congruent and have common radius r. Find r in terms of a and the length $|IO|$.

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The author considers that the problem essentially asks to express r in terms of a and b for $|IO| = 2\sqrt{ab}$. However b could not be used in the problem because of the absence of semicircle β in the figure.

A problem considering a similar figure in a special case $a = b$ can be found in a sangaku in Fukushima, which is supposed to be hung in early years of the Meiji era (1868-1912). The proposer of this problem is Suzuki (鈴木覚治直延) [2].

Circles of radius $r_{\rm A} = ab/(a+b)$ are said to be Archimedean. In this note we give a condition that the figure in the problem can be constructed, and show the existence of two pairs of two congruent non-Archimedean circles. For more recent studies on the arbelos in Wasan geometry see [3, 4].

2. The condition

We consider with a rectangular coordinate system with origin O such that the farthest point on α from AB has coordinates (a, a) (see Figure 3). The point I has coordinates $(0, 2\sqrt{ab})$. The point of intersection of the line AI and α is denoted by P, which has coordinates $(2r_{\rm A}, 2r_{\rm A}\sqrt{a/b})$.

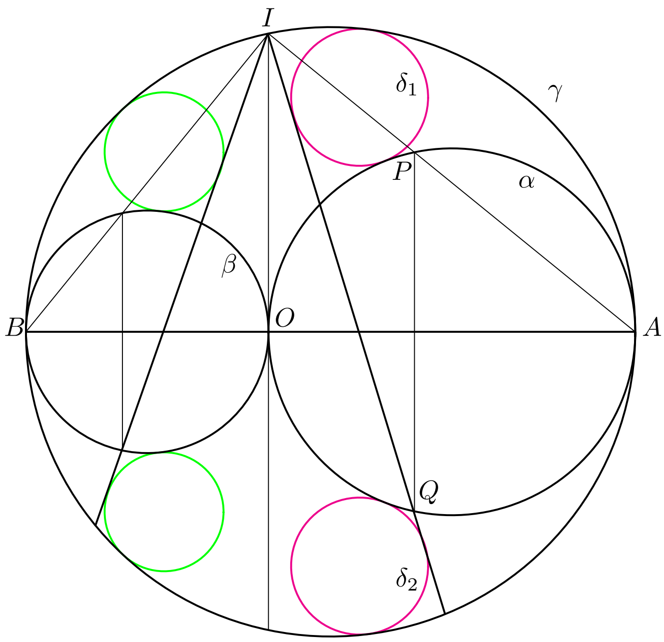

Figure 3.

Theorem 1. *For a point Q on the reflection of α in the line AB, let δ_1 be the incircle of the curvilinear triangle made by α, γ and the line IQ, and let δ_2 be the circle touching the reflections of α and γ in AB and IQ from the side opposite to δ_1. Then δ_1 and δ_2 have common radius r if and only if PQ is parallel to the axis. In this event we have*

$$r = \frac{4a^2b}{(2a+b)^2}. \tag{1}$$

Proof. Let r_i be the radius of δ_i. Then r_1 is maximal if $Q = O$ and minimal if $Q = A$, while the circle δ_2 coincides with the line IO and $r_2 = 0$ in the same cases, respectively. Since r_i changes continuously when the point Q moves from a point close to O to A, there is a case $r_1 = r_2$. We now assume that $r_1 = r_2 = r$. Then their centers have the same x-coordinate. Let $(s, \pm t)$ $(t > 0)$ be the coordinates of the centers. Since IQ passes through the midpoint of the segment joining the two centers, it has an equation

$$2\sqrt{ab}(x - s) + sy = 0. \tag{2}$$

The distances from the center of δ_i to the centers of α, γ and the line IQ are $a + r$, $a + b - r$ and r, respectively. Hence we get $(s - a)^2 + t^2 = (a + r)^2$, $(s - (a - b))^2 + t^2 = (a + b - r)^2$ and $st/\sqrt{4ab + s^2} = r$. Solving the equations, we get (1) and

$$s = \frac{2ab}{2a + b}.$$

In this event the line expressed by (2) passes through the reflection of P in AB, i.e., PQ is parallel to the axis. Since this case is unique, the converse holds. □

Also we get $t = 4a\sqrt{ab(a + b)(4a + b)}/(2a + b)^2$ from the three equations. In the event of the theorem we get two non-Archimedean congruent circles. Exchanging the roles of α and β, we get another pair of two non-Archimedean congruent circles of radius $4ab^2/(a + 2b)^2$, which are indicated by the green circles in Figure 3.

References

[1] Akita (秋田義蕃) ed., Sampō Tenzan Tebikigusa Furoku (算法点竄手引草附録) 1833, Tohoku University Digital Collection,
https://www.i-repository.net/il/meta_pub/G0000398tuldc_4100005576.

[2] Fukushimaken Wasan Kenkyū Hozonkai (福島県和算研究保存会) ed., The Sangaku in Fukushima (福島の算額), Sōju Shuppan (蒼樹出版), 1989.

[3] H. Okumura, The arbelos in Wasan geometry, Tamura's problems, Glob. J. Adv. Res. Class. Mod. Geom., **9** (2019) 33–36.

[4] H. Okumura, The arbelos in Wasan geometry, problems of Izumiya and Naitō, J. Classical Geom., to appear.