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The arbelos in Wasan geometry, the inscribed semicircle in the arbelos

Hiroshi Okumura
Maebashi Gunma 371-0123, Japan
e-mail: hokmr@yandex.com

Abstract. We define an inscribed semicircle in the arbelos, and consider two problems in Wasan geometry, one of which involves an inscribed semicircle in the arbelos. We show that an inscribed semicircle in the arbelos is uniquely determined with its construction and show several properties of it. We also consider three semicircles congruent to the inscribed semicircle in the arbelos and several Archimedean circles.

Keywords. arbelos, inscribed semicircle in the arbelos, Archimedean circle.

Mathematics Subject Classification (2010). 01A27, 51M04, 51M15

1. Introduction

It is little or none to see figures involving semicircles lying inside of a circle. But it is not rare to see such figures in Wasan geometry. We can even see a Wasan book in which all the problems are involving such a semicircle [1]. Let us consider an arbelos formed by the three semicircles α, β and γ of diameters AO, BO and AB, respectively for a point O on the segment AB. We consider a semicircle such that it touches α and β externally and has diameter whose endpoints lie on γ (see Figures 1 and 4). In this case the semicircle is said to be inscribed in the arbelos. We will show that such a semicircle is uniquely determined with its construction, and consider three semicircles congruent to the inscribed semicircle in the arbelos and related Archimedean circles.

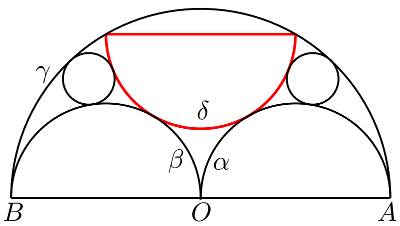

Figure 1.

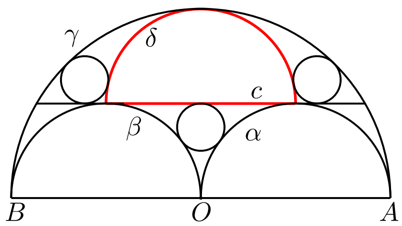

Figure 2.

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We generalize two problems in [7], one of which involves an inscribed semicircle in the arbelos. Let $a = |AO|/2$, $b = |BO|/2$, and let c be the external common tangent of α and β. The problems are stated as follows (see Figures 1 and 2).

Problem 1. Assume $a = b$ and δ is an inscribed semicircle in the arbelos with diameter parallel to AB. Show that the inradius of the curvilinear triangle made by α, γ and δ equals $3a/11$.

Problem 2. Assume $a = b$ and let δ be the maximal semicircle touching γ internally with diameter on c. Show that the radius of γ equals eight times the inradius of the curvilinear triangle made by γ, δ and c.

The incircle of the curvilinear triangle made by α, β and c in Figure 2 is not referred in Problem 2, but we will see that it also congruent to the other two small circles in the case $a = b$.

2. A generalization of Problem 1

In this section we generalize Problem 1. We use the next proposition (see Figure 3).

Proposition 1. *Let C_i $(i = 1, 2)$ be a circle of radius r_i such that $r_1 \neq r_2$. If s is the distance between the centers of the two circles, then the endpoints of a diameter of C_1 lie on C_2 if and only if $s^2 + r_1^2 = r_2^2$.*

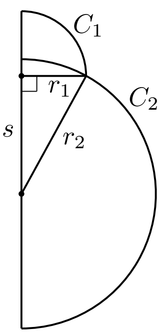

Figure 3.

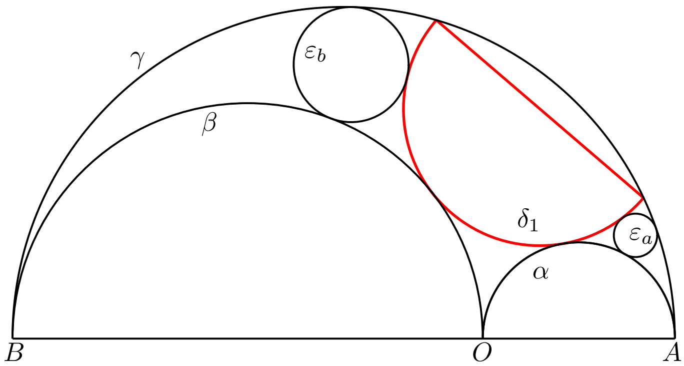

Figure 4.

Let δ_1 be an inscribed semicircle in the arbelos. Let ε_a (resp. ε_b) be the incircle of the curvilinear triangle made by α (resp. β), γ and δ_1 (see Figure 4). We use a rectangular coordinate system with origin O such that the farthest point on α from AB has coordinates (a, a). The radius of Archimedean circles is denoted by r_A, i.e., $r_A = ab/(a + b)$. Let $w = \sqrt{a^2 + 4ab + b^2}$.

Theorem 1. *Let e_a and e_b be the radii of ε_a and ε_b, respectively. Then the following statements are true.*
(i) *The semicircle δ_1 is uniquely determined and has radius $2r_A$.*

(ii) $e_a = \dfrac{abw^2}{aw^2 + 8b^2(a + b)}$. (iii) $e_b = \dfrac{abw^2}{bw^2 + 8a^2(a + b)}$.

Proof. Let δ_1 have center of coordinates (x_1, y_1) and radius d. We get

$$(x_1 - (a - b))^2 + y_1^2 + d^2 = (a + b)^2,$$

$$(x_1 - a)^2 + y_1^2 = (a + d)^2, \quad (x_1 + b)^2 + y_1^2 = (b + d)^2$$

by Proposition 1. Solving the three equation, we get

$$(x_1, y_1) = \left(\frac{2r_{\rm A}(b-a)}{a+b}, \frac{2\sqrt{2}r_{\rm A}w}{a+b} \right), \quad d = 2r_{\rm A}. \tag{1}$$

Therefore δ_1 has unique center and unique radius $2r_{\rm A}$. Hence (i) is proved. We prove (ii). Let (x_a, y_a) be the coordinates of the center of ε_a. Then we have

$$(x_a - x_1)^2 + (y_a - y_1)^2 = (d + e_a)^2, \quad (x_a - a)^2 + y_a^2 = (a + e_a)^2,$$
$$(x_a - (a - b))^2 + y_a^2 = (a + b - e_a)^2.$$

Eliminating x_a and y_a from the three equations and solving the resulting equations for e_a with (1), we get (ii). The part (iii) is obtained similarly. □

We can now call δ_1 *the inscribed semicircle* in the arbelos or *the insemicircle* of the arbelos. The theorem shows that $e_a = e_b = 3a/11$ if $a = b$. Figure 5 is obtained from Figure 1, where the circles, the semicircles and the horizontal lines are denoted by red, green and black, respectively.

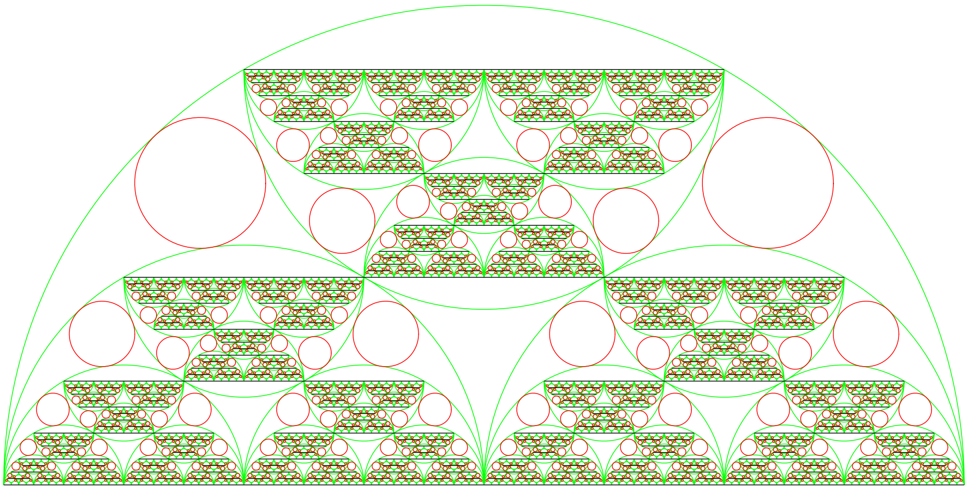

Figure 5.

3. A construction of δ_1

In this section we consider a construction of the semicircle δ_1. Let C and D be the centers of the semicircles γ and δ_1, respectively. Since δ_1 has radius $2r_{\rm A}$, we get $|CD|^2 + (2r_{\rm A})^2 = (a + b)^2$ (see Figure 6). Therefore if R is a point on γ such that $|RS| = 2r_{\rm A}$, where S is the foot of perpendicular from R to AB, then $|CS| = |CD|$ holds. Let ι be the semicircle of center C passing through S constructed on the same side of AB as γ. Then D lies on ι.

The radical axis of α and β is called the axis. Let E be the point of intersection of c and AB in the case $a \neq b$. The point R can be constructed as follows. Let P be the point of intersection of the axis and the line joining the center of β and the farthest point on α from AB. Then $|OP|/b = a/(a + b)$, i.e., $|OP| = r_{\rm A}$. Hence if Q is the reflection of the point O in the point P, then the line parallel to AB passing through Q meets γ in two points whose distance from AB equals $2r_{\rm A}$. Therefore we can choose one of the two points as R. Since $|DE| = |EQ| = 2r_{\rm A}\sqrt{2(a^2 + b^2)}/|a - b|$, D is the point of intersection of the semicircle ι and the circle of center E passing through Q. Notice that the last circle and ι are orthogonal, since CD is perpendicular to DE. Also notice that the circles of diameters OQ and RS are Archimedean.

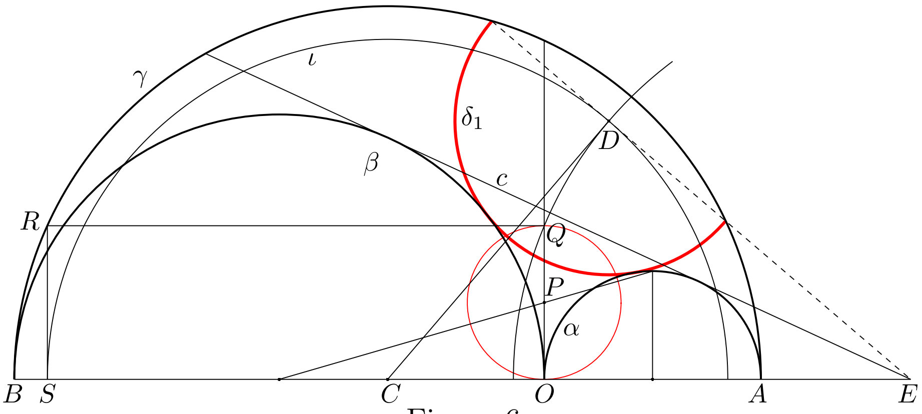

Figure 6.

Now the semicircle δ_1 can be constructed in the following way. Let P be the point of intersection of the axis and the line joining the center of β and the farthest point on α from AB. Let Q be the reflection of O in P. Let R be one of the points of intersection of γ and the line passing through Q parallel to AB, and let S be the foot of perpendicular from R to AB. Draw the semicircle ι of center C passing through S lying on the same side of AB as γ. Then D is the point of intersection of ι and the circle of center E passing through Q. If $a = b$, then D is the point of intersection of ι and the axis. Then δ_1 is the semicircle of radius $|OQ|$ and center D.

4. A generalization of Problem 2

In this section we generalize Problem 2. The point of intersection of γ and the axis is denoted by I. Considering the power of the point O with respect to γ, we see that I has coordinates $(0, 2\sqrt{ab})$. Let δ_2 be the maximal semicircle touching γ internally and having diameter on c. Since the maximal circle touching γ internally and c has radius r_{A} [2], δ_2 has radius $2r_{\mathrm{A}}$. The line c has the following equation [6], [4]:

$$(a-b)x - 2\sqrt{ab}y + 2ab = 0. \tag{2}$$

Therefore the distance form I to c equals $2r_{\mathrm{A}}$. This implies that δ_2 touches γ at I, and the center of δ_2 coincides with the point of intersection of c and the line CI. Hence it has coordinates

$$(x_2, y_2) = \left(\frac{2r_{\mathrm{A}}(a-b)}{a+b}, \frac{2\sqrt{ab}(a^2+b^2)}{(a+b)^2} \right). \tag{3}$$

Let ε_1 be the incircle of one of the curvilinear triangles made by γ, δ_2 and c (see Figure 7). Let ε_2 be one of the circles touching γ or the reflection of γ in AB internally, the reflection of δ_2 in c externally, and c from the side opposite to ε_1.

Theorem 2. *Let e_i be the radius of the circle ε_i. The following statements hold.*
(i) *The semicircle δ_2 has radius $2r_{\mathrm{A}}$.*

(ii) $e_1 = \dfrac{ab(a^2+b^2)}{(a+b)^3}$. (iii) $e_2 = \dfrac{a^2+b^2}{2(a+b)}$. (iv) $\dfrac{1}{e_1} - \dfrac{1}{e_2} = \dfrac{1}{r_{\mathrm{A}}}$.

Proof. We have already proved (i). If the circles ε_1 or ε_2 has radius e and center of coordinates (p, q), we get

$$(p - x_2)^2 + (q - y_2)^2 = (2r_{\rm A} + e)^2, \quad (p - (a - b))^2 + q^2 = (a + b - e)^2,$$

$$\frac{|(a - b)p - 2\sqrt{ab} \cdot q + 2ab|}{a + b} = e$$

by (2). Eliminating p and q from the three equations and solving the resulting equation for e with (3), we have

$$e = \frac{ab(a^2 + b^2)}{(a + b)^3}, \quad e = \frac{a^2 + b^2}{2(a + b)}.$$

Since the latter is larger the the former, we get (ii) and (iii). The part (iv) follows from (ii) and (iii). □

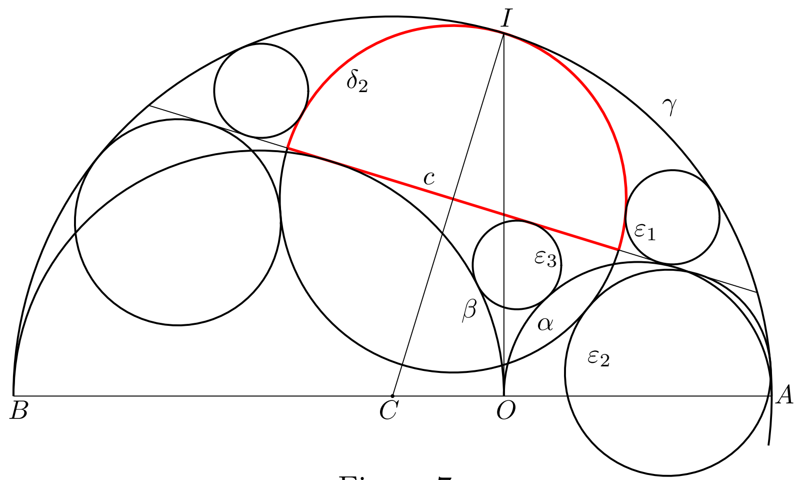

Figure 7.

From Figure 2, we can construct a Sierpiński-gasket-like recursive configuration as in Figure 8, where the circles, the semicircles and the horizontal lines are denoted by red, green and black, respectively.

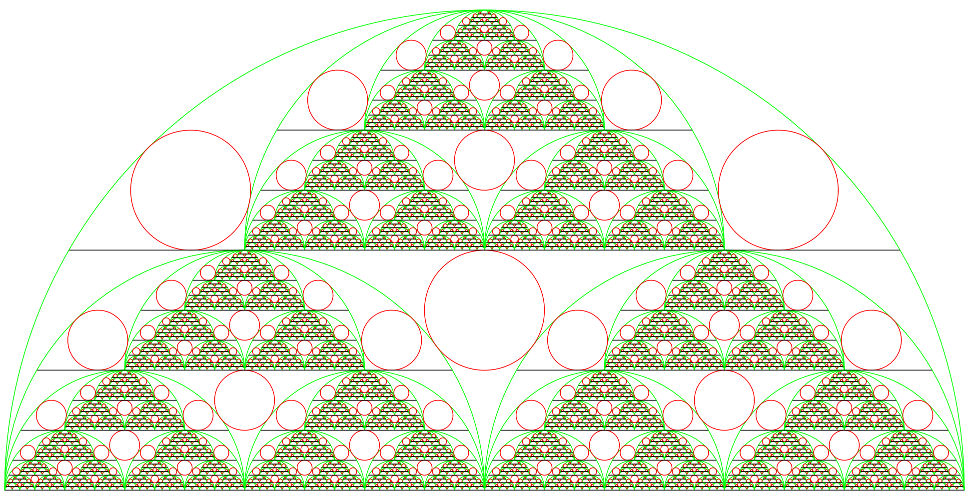

Figure 8.

5. Two semicircles of radius $2r_{\rm A}$

In this section we consider the semicircles δ_1 and δ_2 together (see Figure 9).

Theorem 3. *The following statements hold.*
(i) *The semicircles δ_1 and δ_2 are congruent and have radius $2r_{\rm A}$.*
(ii) *The extended diameters of δ_1 and δ_2 are parallel to AB or meet in E.*
(iii) *If J is the point of intersection of the axis and the line CD, then the circle of diameter JO is orthogonal to δ_1. Therefore this circle passes through the points of tangency of two of α, β and δ_1.*

Proof. The part (i) follows from Theorems 1 and 2. The part (ii) is trivial if $a = b$. We assume $a \neq b$. Let ζ be the circle of center E passing through O. If we invert the figure in the circle of center O passing through I, then α, β and ζ are inverted into three lines l_a, l_b and l_z expressed by $x = 2b$, $x = -2a$ and $x = b - a$, respectively, because $|IO|^2 = 4ab$. Since any proper circle touching l_a and l_b are orthogonal to l_z, any circle touching α and β at points different from O is orthogonal to ζ, i.e., such a circle is fixed by the inversion in ζ. Hence γ and δ_1 are fixed by the inversion in ζ and the points of intersection of γ and δ_1 are interchanged by the inversion. This proves (ii). The point J has coordinates $(0, 2y_j)$ by (1), where $y_j = \sqrt{2}ab/w$. Hence the circle of diameter JO has center of coordinates $(0, y_j)$ and radius $r_j = y_j$. Then $(x_1 - 0)^2 + (y_1 - y_j)^2 - (2r_{\rm A})^2 - r_j^2 = 0$ holds. This proves (iii). □

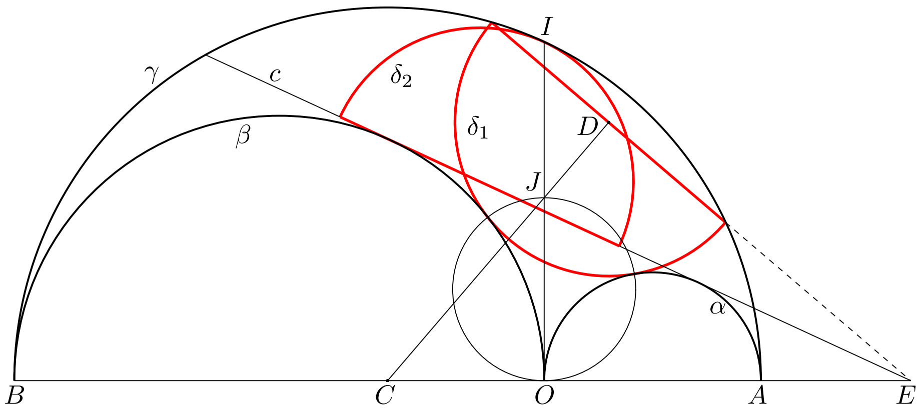

Figure 9.

6. More on semicircle of radius $2r_{\rm A}$ and Archimedean circles

We have considered the circle of radius $2r_{\rm A}$ touching γ and the line AB at the point O in [6]. Hence we can get a semicircle of radius $2r_{\rm A}$ such that it has diameter on the axis and touches γ and AB at O. In this section we consider this semicircle a bit further together with its counterpart and related Archimedean circles. Let τ_1 be the circle touching α, β and c. Let τ_2 be the circle touching the reflections of α and β in AB and c. We denote the coordinates of the center of τ_i by (p_i, q_i), and its radius by t_i.

Proposition 2. *If $a \neq b$, the following relations hold.*

(i) $p_1 = r_{\rm A}\left(1 - \dfrac{2\sqrt{a}}{\sqrt{a}+\sqrt{b}}\right)$, $q_1 = 2r_{\rm A}\left(1 - \dfrac{\sqrt{ab}}{(\sqrt{a}+\sqrt{b})^2}\right)$, $t_1 = \dfrac{ab}{(\sqrt{a}+\sqrt{b})^2}$.

(ii) $p_2 = r_{\rm A}\left(1 - \dfrac{2\sqrt{a}}{\sqrt{a}-\sqrt{b}}\right)$, $q_2 = -2r_{\rm A}\left(1 + \dfrac{\sqrt{ab}}{(\sqrt{a}-\sqrt{b})^2}\right)$, $t_2 = \dfrac{ab}{(\sqrt{a}-\sqrt{b})^2}$.

Proof. If (p, q) are the coordinates of the center of τ_i and t is its radius, we have

$$(p-a)^2+q^2=(a+t)^2, \quad (p+b)^2+q^2=(b+t)^2,$$

$$\frac{|(a-b)p-2\sqrt{ab}\cdot q+2ab|}{a+b}=t.$$

Solving the equations for p, q and t, we get (i) and (ii). □

Proposition 2(i) holds in the case $a = b$. In this case the three small circles in Figure 2 are congruent, since $e_1 = t_1 = a/4$ by Theorem 2(ii). On the other hand (ii) shows that division by zero occurs in this case. We hope to consider (ii) in this case in a later paper by division by zero calculus [5].

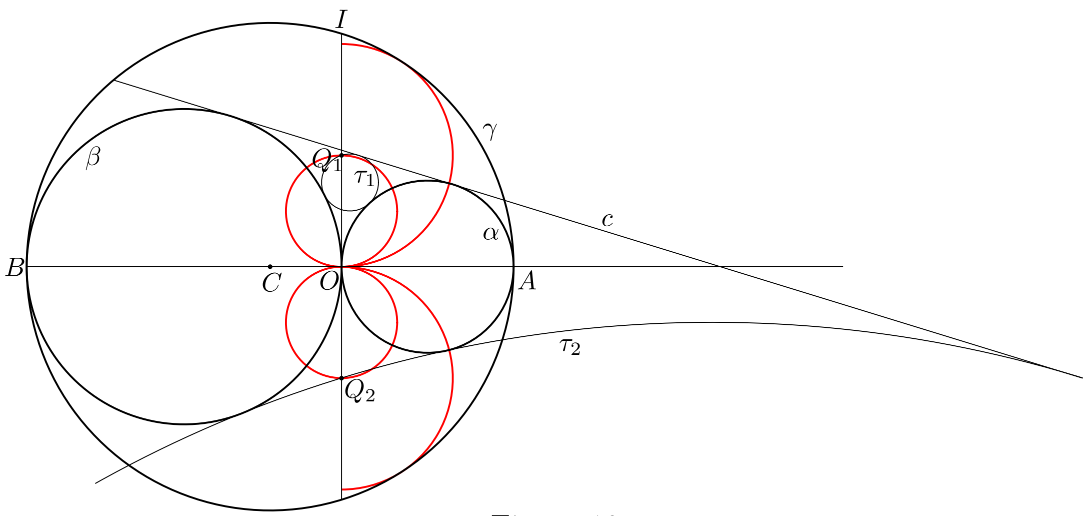

Figure 10.

Theorem 4. *Let Q_i be the point of intersection of the axis and the circle τ_i closer to the point I for $i = 1, 2$. The following statements hold.*
(i) *The circle of diameter OQ_i is Archimedean.*
(ii) *If $a \neq b$, there is a unique semicircle of radius $2r_{\rm A}$ and center Q_i such that it has diameter on the axis and touches γ and the line AB at O.*
(iii) *The smallest circle of center on τ_1 and touching AB also touches c and is Archimedean.*
(iv) *If $a \neq b$, the smallest circle of center on τ_2 and touching AB also touches c and is Archimedean.*

Proof. If $a = b$, τ_i is a line parallel to AB, where the distance between τ_i and AB equals $a = 2r_{\rm A}$. Therefore (i) holds. Assume $a \neq b$. By Proposition 2(ii), the circle τ_2 meets the axis in the points of coordinates

$$(0, -2r_{\rm A}) \;\; and \;\; \left(0, -\frac{2ab}{(\sqrt{a}-\sqrt{b})^2}\right).$$

While $-2r_{\rm A} - \left(-\frac{2ab}{(\sqrt{a}-\sqrt{b})^2}\right) = \frac{4r_{\rm A}\sqrt{ab}}{(\sqrt{a}-\sqrt{b})^2} > 0$. Hence the point Q_2 has coordinates $(0, -2r_{\rm A})$ (see Figure 10). This proves (i) for $i = 2$. The case $i = 1$ is proved similarly. Since Q_1 and Q_2 have coordinates $(0, 2r_{\rm A})$ and $(0, -2r_{\rm A})$, respectively, we have $|CQ_1| = |CQ_2| = a + b - 2r_{\rm A}$. This proves (ii). Since $q_1 - t_1 = r_{\rm A}$, the closest point on τ_1 to AB has coordinates $(p_1, r_{\rm A})$, while $((a -$

$b)p_1 - 2\sqrt{ab} \cdot r_A + 2ab)/(a+b) = r_A$ (see Figure 11). This proves (iii). The part (iv) is proved in a similar way. □

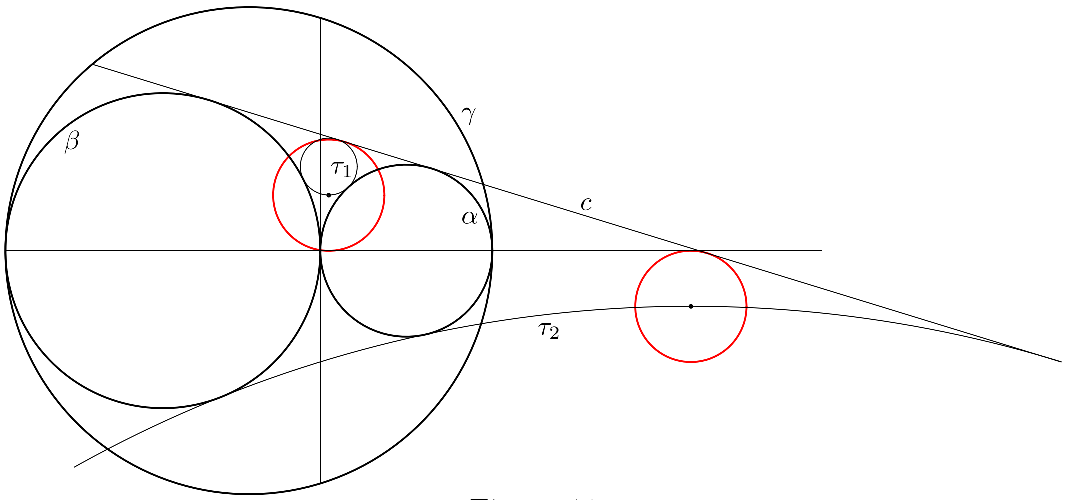

Figure 11.

If $a = b$, there are two semicircles of radius $2r_A$ satisfying (ii), both touch γ at the point I. The Archimedean circle of diameter OQ_1 was found by L. Bankoff [3]. Theorem 4(i) shows that the circle of diameter OQ_2 is the counterpart of this circle.

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