Sangaku Journal of Mathematics (SJM) ©SJM
ISSN 2534-9562
Volume 7 (2023), pp.13-15
Received 27 February 2023. Published on-line 29 March 2023
web: http://www.sangaku-journal.eu/

Note on a Sangaku like construction of $X(55)$

Paris Pamfilos
Estias 4, 71307 Heraklion, Greece
e-mail: pamfilos@uoc.gr

Abstract. We present an elementary construction, reminiscent of a Sangaku configuration, of the triangle center $X(55)$, known to be the insimilicenter of the circumcircle and the incircle of the triangle.

Keywords. triangle center, $X(55)$.

Mathematics Subject Classification (2020). 51-02, 51M15.

1. Introduction

The *"triangle center"* $S = X(55)$ is known ([1]) to be the inner similarity center of the incircle $\kappa(I, r)$ and the circumcircle $\kappa'(O, R)$ of the triangle ABC (see Figure 1). As suggested by the figure, we present an elementary proof that S

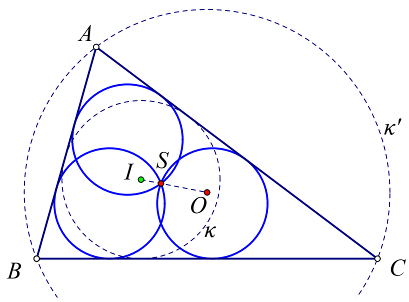

Figure 1. The triangle center $S = X(55)$

coincides with the common point of three equal circles, each tangent to two sides of the triangle.

2. Construction of S

We start with a triangle $A''B''C''$, similar to the given one ABC, and construct three circles equal to its circumcircle but with centers at its vertices (see Figure 2). Obviously the three circles pass through the circumcenter S of $A''B''C''$. Then, we draw the common tangents to the three circles defining the triangle $A'B'C'$, which obviously has sides parallel to those of $A''B''C''$ hence is homothetic to it.

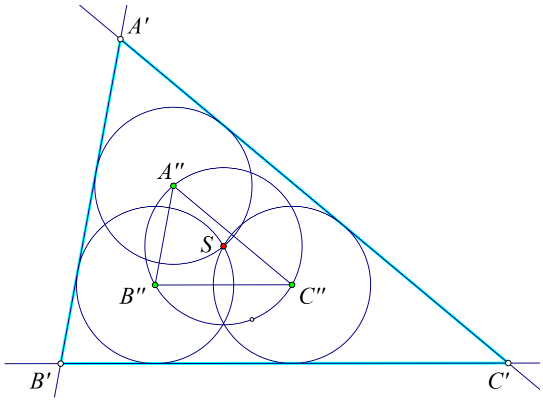

FIGURE 2. Construction of $A'B'C'$ homothetic to $A''B''C''$

Theorem 2.1. *The point S is the inner similarity center of the incircle and the circumcircle of $A'B'C'$.*

Proof. Consider the circumcircle $\lambda(O')$ of $A'B'C'$ and repeat the preceding construction of circles equal to λ at the vertices of $A'B'C'$. Consider one of these circles, λ_C say, centered at C' (see Figure 3). Obviously $C'O'$ and $C''S$ are par-

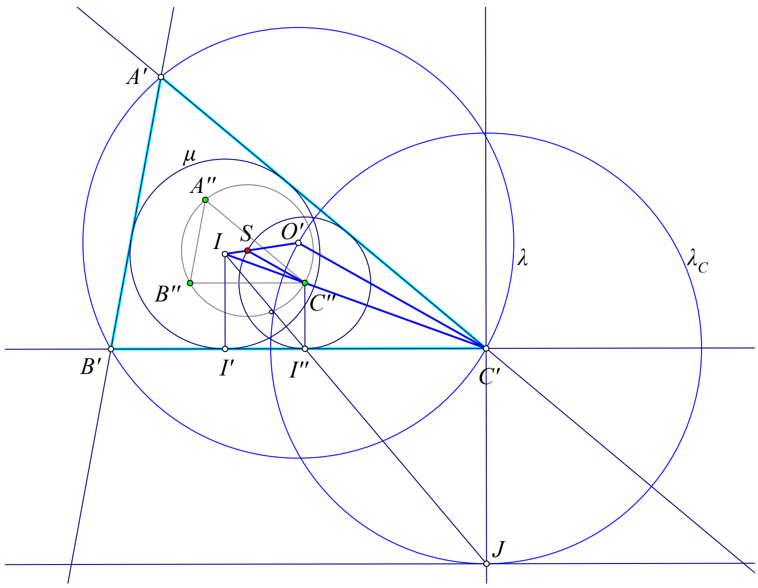

FIGURE 3. Similarity center I of circles $S(|SC''|)$ and $C'(|C'J|)$

allel and $C'C''$ is the inner bisector of the angle $\widehat{C'}$ meeting line $O'S$ at a point I. Analogously the lines $B'B''$ and $A'A''$ will meet also at I on the line $O'S$. But

I is obviously the incenter of $A'B'C'$. It follows that S is the inner similarity center of the circle λ with the incircle μ of $A'B'C'$. In fact, project points I and C'' on line $B'C'$ and draw the tangent to λ_C at J parallel to $B'C'$. The lengths of the segments II' and $C'J$ are correspondingly the radii of the incircle and the circumcircle λ of $A'B'C'$. Their ratio is equal to $II''/I''J$ which by the parallels transfers to $IC''/C''C' = IS/SO'$, thereby proving the theorem. □

Since the characteristic property of S is preserved by similarities, we can transfer the preceding construction using a similarity mapping $A'B'C'$ to the given triangle ABC.

References

[1] K. Kimberling, Encyclopedia of Triangle Centers, `https://faculty.evansville.edu/ck6/encyclopedia/ETC.htmll`.