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Incorrect sangaku problems involving circles in a rectangle

Hiroshi Okumura
Maebashi Gunma 371-0123, Japan
e-mail: hokmr@yandex.com

Abstract. We consider three incorrect sangaku problems involving circles in a rectangle.

Keywords. rectangle, square, 3-4-5 triangle

Mathematics Subject Classification (2010). 01A27, 51M04

1. A PROBLEM INVOLVING THREE CONGRUENT CIRCLES IN A RECTANGLE

The following problem was recorded in 1803 by Furukawa (古川 [王君] 童) ([3]) (see Figure 1).

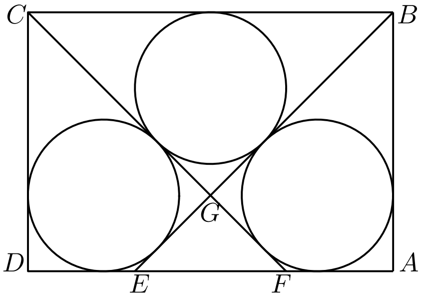

Figure 1.

Problem 1. For a rectangle $ABCD$, E and F are points on the side DA such that the segments BE and CF meet in a point G lying inside of $ABCD$ and the quadliraterals $ABGF$ and $CDEG$ and the triangle BCG have congruent incircles. If $|AB| = 3$ and $|BC| = 6$, find the radius of the three congruent circles.

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1.1. **Solution.** Since the incircles of the two quadrilaterals are congruent, the figure is symmetric in the perpendicular bisector of the side BC. Therefore BCG is an isosceles triangle with $|BG| = |CG|$. Assume that the lines AB and CF meet in a point H (see Figure 2). We have $\angle GBH + \angle CBG = \pi/2 = \angle BHC + \angle HCB$, while $\angle CBG = \angle HCB$. This implies $\angle GBH = \angle BHC$. Hence the triangle BGH is isosceles with $|BG| = |GH|$. Let r be the radius of the congruent circles. Also let $w = |BC|$, $v = |BG|$ and $\theta = \angle BGC$. The areas of the triangles BCG and BGH equal $(v^2 \sin\theta)/2$. While they also equal $(2v + w)r/2$ and $(2v + |BH|)r/2$, respectively. Hence we have $w = |BH|$. Therefore the two triangles are congruent. Hence $\angle CBG = \pi/4$ and $r/(w/2) = \tan\pi/8 = \sqrt{2} - 1$. Therefore

$$r = \frac{\sqrt{2}-1}{2}w.$$

Also we have

$$|AB| = \frac{w}{2} + r = \frac{w}{\sqrt{2}}. \tag{1}$$

(1) shows that $|AB| = 3$ implies $|BC| = 3\sqrt{2}$. Therefore the problem is incorrect. The above solution shows that BCH, BGH and BCG are isosceles right triangles. The figure in the problem can be embedded in a tiling pattern. Figure 3 shows such an example.

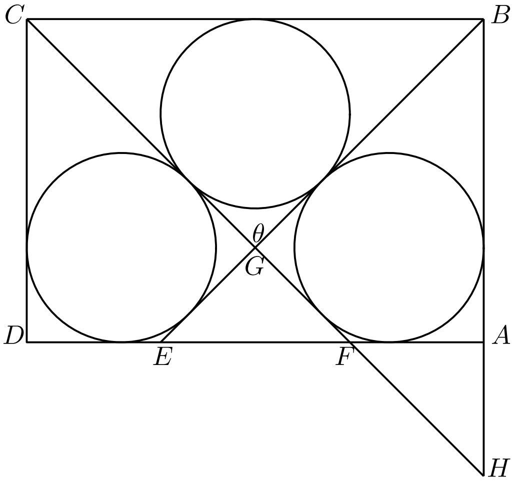

Figure 2.

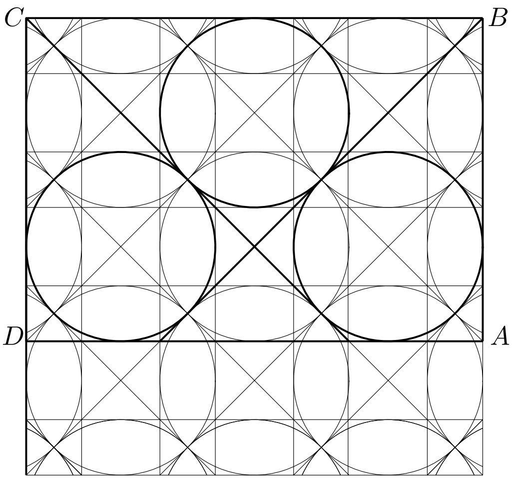

Figure 3.

2. A PROBLEM INVOLVING THREE CIRCLES IN A RECTANGLE

The next problem was proposed by San Ou or San Nou ([髟參] 翁) in 1811 ([1]) (see Figure 4).

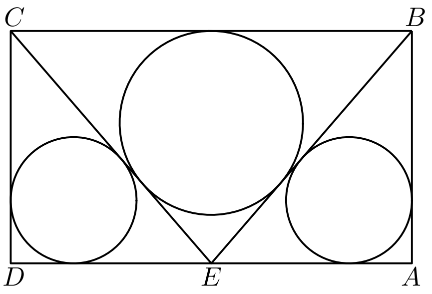

Figure 4: The figure $\mathcal{F}$.

Problem 2. Let E be the midpoint of the side DA for a rectangle $ABCD$. The product of $|AB|$ and $|BC|$ equals 336, and the product of the inradii of the triangles ABE and BCE equals 63/4. Find the inradius of the triangle BCE.

The proposer gives an answer 5.25. However we show that there is another solution later.

2.1. **Solution.** The figure in the problem is denoted by $\mathcal{F}$. Let $|AB| = h$, $|BC| = w$ for $\mathcal{F}$. The inradii of the triangles BCE and ABE are denoted by r_1 and r_2, respectively (see Figure 5).

We give a general solution. Let $\theta_1 = \angle CBE/2$ and $\theta_2 = \angle EBA/2$. Since the segments joining the point B and the incenters of BCE and ABE are the angle bisectors of $\angle CBE$ and $\angle EBA$, we have $\theta_1 + \theta_2 = \pi/4$. Therefore we have $\tan(\theta_1 + \theta_2) = 1$, where

$$\tan\theta_1 = \frac{r_1}{w/2} \quad \textit{and} \quad \tan\theta_2 = \frac{r_2}{h - r_2}. \tag{2}$$

Substituting (2) in $\tan\theta_1 + \tan\theta_2 = 1 - \tan\theta_1 \tan\theta_2$ and rearranging, we have

$$2hr_1 + w(2r_2 - h) = 0. \tag{3}$$

On the other hand, from $\angle AEB = \angle CBE$, we get

$$\frac{r_2}{w/2 - r_2} = \frac{r_1}{w/2}.$$

Hence we have

$$w(r_1 - r_2) - 2r_1r_2 = 0. \tag{4}$$

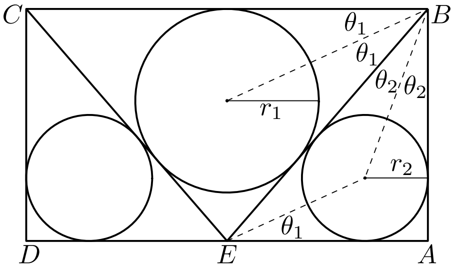

Figure 5.

Let $U = r_1r_2$ and $V = hw$. Eliminating w, h and r_2 from the four equations (3), (4), $r_1r_2 = U$ and $hw = V$ and rearranging, we get

$$Vr_1^4 - 3UVr_1^2 + 2U^2(2U + V) = 0. \tag{5}$$

Considering this as a quadratic equation of r_1^2 and solving we get

$$r_1^2 = \frac{U}{2}\left(3 \pm \sqrt{1 - \frac{16U}{V}}\right), \tag{6}$$

where the discriminant of (5) equals $U^2V(V - 16U)$. Hence (5) has no real root if $V < 16U$. If $V = 16U$, then (5) has a positive multiple root:

$$r_1^2 = \frac{3U}{2}. \tag{7}$$

Assume $V > 16U$. Since $3^2 - \left(\sqrt{1-16U/V}\right)^2 = 8 + 16U/V > 0$. (5) has two positive roots. We now get a general solution of the problem:

$$r_1 = \sqrt{\frac{U}{2}\left(3 \pm \sqrt{1-\frac{16U}{V}}\right)} \text{ in the case of } V \geq 16U. \tag{8}$$

Problem 2 assumes $U = 63/4$ and $V = 336$, i.e., $V > 16U$. Therefore (8) shows that r_1 has two positive values. And we get $r_1 = 5.25$ and $r_1 = 4.43\cdots$ by (8). Hence the proposer of the problem missed the latter case. If $r_1 = 5.25$, then we have $r_2 = U/r_1 = 3$, $w = 14$ by (4) and $h = V/w = 24$. Figure 6 shows the figure in this case. If $r_1 = 4.43\cdots$, then $r_2 = 3.54\cdots$, $w = 35.49\cdots$ and $h = 9.46\cdots$. Figure 7 shows the figure in this case.

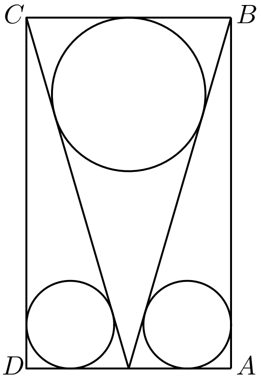

Figure 6.

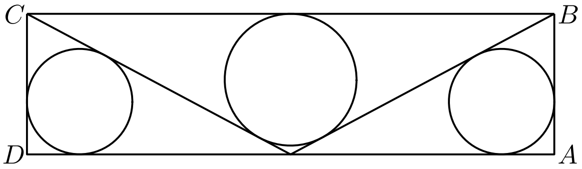

Figure 7.

2.2. **Overlapping case.** (8) shows that *two figures exist if* $V > 16U$ *and they overlap if* $V = 16U$. Assume $V = 16U$. Then $r_1 = \sqrt{3U/2}$ by (7), $r_2 = \sqrt{2U/3}$, $w = 2\sqrt{6U}$ by (4) and $h = 4\sqrt{2U/3}$. Therefore

$$r_1 = \frac{3}{2}r_2, \quad w = 6r_2, \quad h = 4r_2.$$

Hence the rectangle $ABCD$ is covered by six squares with inradius r_2 (see Figure 8).

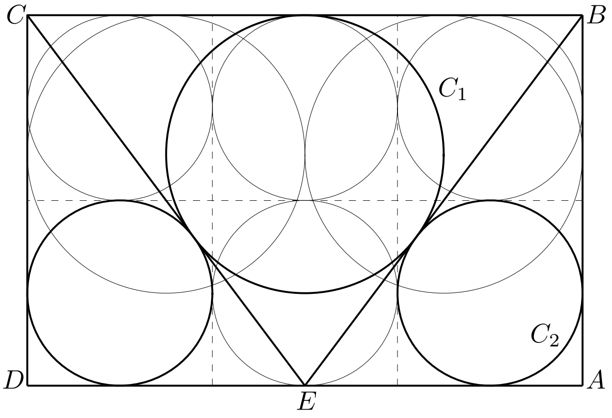

Figure 8.

We denote the incircles of the triangles BCE and ABE by C_1 and C_2, respectively. If the side BE touches the circles C_1 and C_2 at points P_1 and P_2, respectively. Then $|BP_1| = 3r_2 = |BP2|$. Hence $P_1 = P_2$. Therefore *the two circles touch*. Notice that ABE is a 3-4-5 triangle.

Nishiyama (西山弥平次) considered the figure under consideration, and proposed the problem in 1870 to find r_2 in the case $|AB| = 4$, $|BC| = 6$ and $r_1 = 3/2$ for $\mathcal{F}$ ([5]). However the three assumptions can be reduced to two.

2.3. **The case $ABCD$ being a square.** The remaining tangent of the incircles of the triangles ABE and CDE does not touch the circle C_1 in general for $\mathcal{F}$. We consider the case in which they touch. We use the next proposition.

Proposition 1. *If s is the semiperimeter of a triangle with the sides a, b and c, then the inradius equals $\sqrt{(s-a)(s-b)(s-c)/s}$.*

We get the following theorem.

Theorem 1. *For the figure $\mathcal{F}$, the remaining external common tangent of the incircles of the triangles ABE and CDE touches the incircle of the triangle BCE if and only if $ABCD$ is a square.*

Proof. Assume $|BC| = 2$. Then $|BE| = \sqrt{1+h^2}$. Therefore we have

$$(9) \quad r_1 = \frac{\sqrt{2+h^2-2\sqrt{1+h^2}}}{h} \text{ and } r_2 = \frac{\sqrt{2}h}{2\sqrt{1+\sqrt{1+h^2}+h\left(1+h+\sqrt{1+h^2}\right)}}$$

by Proposition 1. The common tangent of the incircles touches C_1 if and only if $2(r_1 + r_2) = h$. Substituting (9) in the last equation and solving the resulting equation for h, we get $h = 2$. □

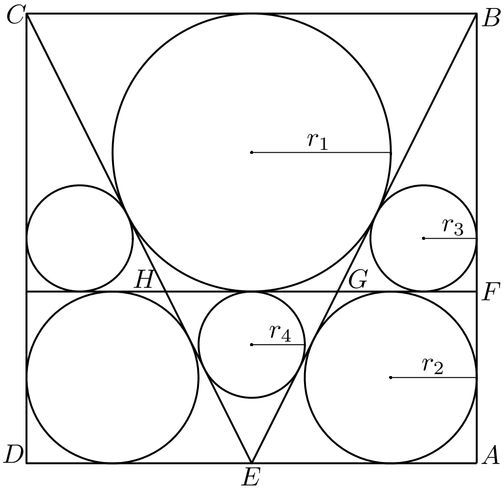

Figure 9.

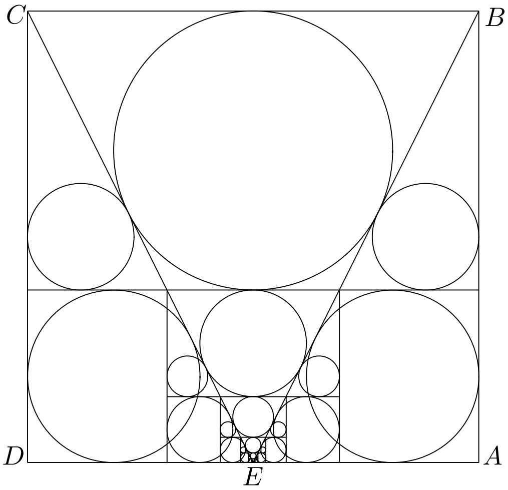

Figure 10.

We consider the case in which $ABCD$ is a square. Assume $|AB| = 2$, and the tangent meets the segments AB, BE and CE in points F, G and H, respectively, and the triangles BGF and EGH have inradii r_3 and r_4, respectively (see Figure 9). Then $|BE| = \sqrt{5}$ and $r_1 = \left(\sqrt{5}-1\right)/2$ and $r_2 = \left(3-\sqrt{5}\right)/2$ by (9). From $|BF| = (\sqrt{5}-1)$ and $|AF| = (3-\sqrt{5})$, we get $r_3 = r_2|BF|/|AB| = (\sqrt{5}-2)$ and $r_4 = r_1|AF|/|AB| = (\sqrt{5}-2)$. Hence *the triangles BGF and EGH have congruent incircles.* Figure 10 shows a recursive figure derived from Figure 9.

Tsuboi (壷井新三郎) considered the problem to find r_1 in terms of $|AB|$ in the case $ABCD$ being a square for the figure $\mathcal{F}$ ([4]).

3. A PROBLEM INVOLVING n CONGRUENT CIRCLES IN A RECTANGLE

The next problem was proposed by Satoh (佐藤善太郎) in 1846 ([6], [7]) (see Figure 11).

Problem 3. For a rectangle $ABCD$, E is a point on the side BC. $\gamma_1, \gamma_2, \cdots, \gamma_n$ are congruent circles of radius r lying inside of the triangle CDE and touching the side CD such that γ_1 touches DE, γ_i touches γ_{i-1} from the side opposite to DA for $i = 2, 3, \cdots, n$, and γ_n touches the side CE. The triangle ABE has inradius r. Find the inradius of the triangle AED.

Let i_r be the inradius of ADE. The answer says

$$i_r = \left(\frac{n}{2} + 1\right) r. \tag{10}$$

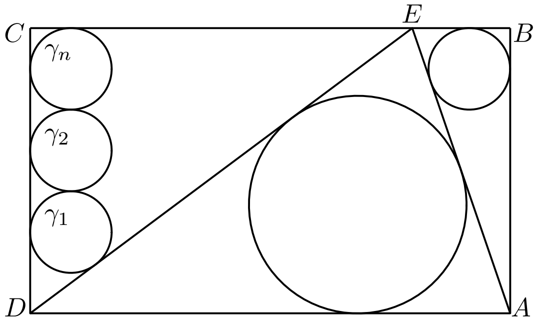

Figure 11.

3.1. **Incorrectness.** We have shown that (10) is correct if $n = 2$ in [2]. We show it is incorrect if $n > 2$. The figure in Problem 3 is called an n-circle rectangle configuration.

Let $\mathcal{R}$ be an n-circle rectangle configuration. The images of the circles γ_1, γ_2, $\cdots$, γ_n by the rotation through angle π in the midpoint of DE are denoted by δ_1, δ_2, $\cdots$, δ_n, and the incircle of the triangle AEF is denoted by δ_0 (see Figure 12). We call the figure consisting of the triangle AED with the point F and the circles δ_0, δ_1, δ_2, $\cdots$, δ_n an n-circle triangle configuration. An n-circle triangle configuration can be derived from an n-circle rectangle configuration and vice versa. Let $\mathcal{T}$ be the n triangle configuration derived from the configuration $\mathcal{R}$.

Let E_0 be a point on the side EF such that the remaining tangent of δ_1 from E_0 is parallel to DA for $\mathcal{T}$. If $E'(\neq E)$ is a point on the half line E_0F with initial point F such that E' is farther from F than E_0, then the remaining tangents of the circles δ_0 and δ_1 from E' meet the line DA. Therefore using the point E' instead of E, we get an n-circle triangle configuration different from $\mathcal{T}$. The figure also derives an n-circle rectangle configuration different from $\mathcal{R}$.

Recall that i_r is the inradius of the triangle AED in Problem 3. We now obtain i_r. We use a Cartesian coordinates system with origin F so that the center of δ_0 has coordinates (r, r). Let $h = |AB|$. The circle δ_0 has an equation $(x-r)^2 + (y-r)^2 = r^2$ and the line AE has an equation $y - h = m_1 x$ for a real number m_1. Eliminating y from the last two equations we get a quadratic equation of x. Let Δ be its discriminant. Then the equation $\Delta = 0$ implies

$$m_1 = -\frac{h(h-2r)}{2(h-r)r}.$$

This is the slope of the line AE. Similarly we get the slope of the line DE:

$$m_2 = \frac{(h-2(n-1)r)(h-2nr)}{2(h-(2n-1)r)r},$$

where δ_1 has an equation $(x+r)^2 + (y-(2n-1)r)^2 = r^2$.

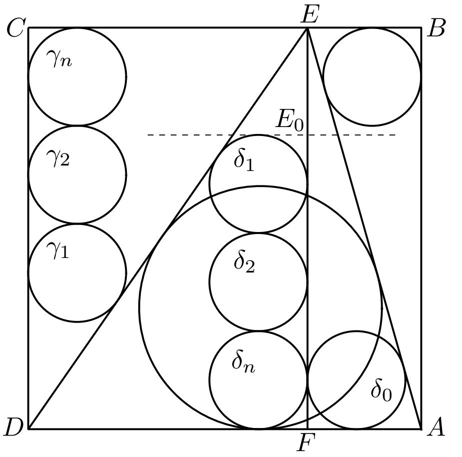

Figure 12: $\mathcal{R}_n$ and $\mathcal{T}_n$ $(n = 3)$.

Let x_1 and x_2 be the x-coordinates of the points A and D, respectively. Then we have $0 - h = m_1 x_1$ and $0 - h = m_2 x_2$. Therefore we have

$$x_1 = \frac{2(h-r)r}{h-2r} \quad \textit{and} \quad x_2 = -\frac{2h(h-(2n-1)r)r}{(h-2(n-1)r)(h-2nr)}.$$

Then we have

$$|AE| = \sqrt{x_1^2 + h^2} = \frac{h^2 - 2hr + 2r^2}{h-2r},$$

$$|DE| = \sqrt{(-x_2)^2 + h^2} = h\sqrt{\frac{4(h-(2n-1)r)^2 r^2}{(h-2(n-1)r)^2(h-2nr)^2} + 1},$$

and

$$|DA| = x_1 - x_2 = 2r\left(\frac{h(h-(2n-1)r)}{(h-2(n-1)r)(h-2nr)} + \frac{h-r}{h-2r}\right).$$

Therefore by Proposition 1, we have

$$i_r = \frac{2r(h-nr)}{h-2(n-1)r}. \tag{11}$$

This is a solution of Problem 3. Then we have

$$\frac{di_r}{dh} = -\frac{2(n-2)r^2}{(h-2(n-1)r)^2}.$$

Therefore $di_r/dh < 0$ if $n > 2$. Hence i_r is a monotonically decreasing function of h if $n > 2$, while (10) asserts that i_r is constant. Therefore (10) is incorrect if $n > 2$.

Since

$$\lim_{h\to\infty} i_r = \frac{2r(1-nr/h)}{1-2(n-1)r/h} = 2r,$$

i_r approaches $2r$ if h increases.

3.2. **Correction of the problem.**

We have

$$i_r - \left(\frac{n}{2}+1\right) r = \frac{(n-2)(h-2(n+1)r)r}{-2(h-2(n-1)r)}$$

by (11). Therefore (10) is correct if and only if $n = 2$ or $h = 2(n+1)r$.

Assume $h = 2(n+1)r$. Then *there is the circle of radius r touching γ_1 and the sides CD and DF*. Conversely the problem is correct if this circle exists. We assume that the circle exists and denote it by ε_1. Since $|CD|/|DF| = h/(-x_2) = 4/3$, CDE is a 3-4-5 triangle. Hence if $1+n = 4j$ for a positive integer j, then there are circles ε_2, ε_3, $\cdots$, ε_{3j} of radius r touching the side DF such that ε_k touches ε_{k-1} from the side opposite to CD for $k = 2, 3, \cdots, 3j$ and ε_{3j} touches the side EF from the same side as ε_1 (see Figure 13). Notice that we have obtained a characterization of the 3-4-5 triangle (see Figure 14):

Theorem 2. *For a right triangle ABC with right angle at C, p is the perpendicular to BC at B. Then ABC is a 3-4-5 triangle with $|BC|/|CA| = 4/3$ if and only if there are two congruent circles γ_1 and γ_2 such that γ_1 touches p and the side BC from the same side as A, and γ_2 touches γ_1 and the sides AB and BC.*

Notice that Figure 8 (see the triangle ABE) implicitly shows this theorem, where the circle γ_2 is the incircle of the triangle, i.e., $\gamma_2 = C_2$.

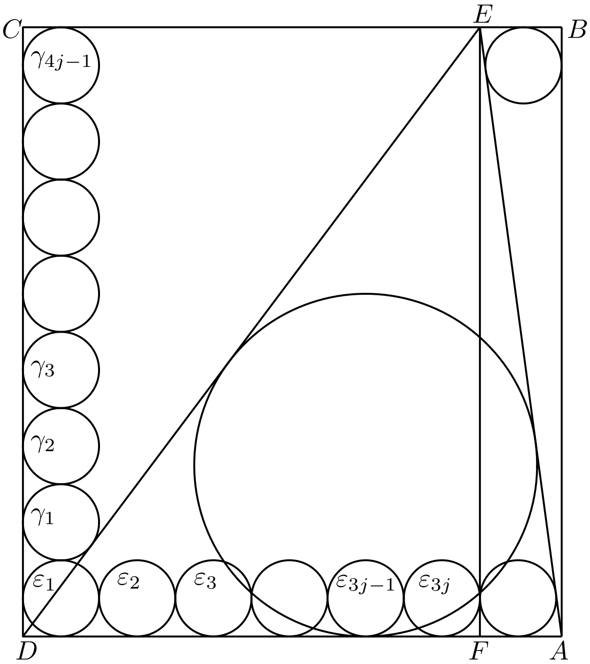

Figure 13: $j = 2$, $n = 7$.

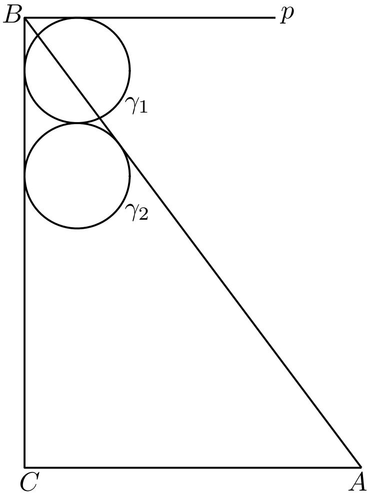

Figure 14.

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