Sangaku Journal of Mathematics (SJM) ©SJM
ISSN 2534-9562
Volume 9 (2025), pp. 9–14
Received 9 June 2025. Published on-line 16 July 2025.
web: https://sangaku-journal.com/

A Brief Note on a Misleading Problem Figure

J. MARSHALL UNGER

3144 Gracefield Rd., Apt. T-09, Silver Spring, MD (USA)
e-mail: unger.26@osu.edu

Abstract. Aida Yasuaki provides a brilliant algebraic solution for the first problem presented in [1], but a key figure he provides to explain it is misleading, making it appear that three points are collinear even though they need not be.

Keywords. *wasan*, sagitta, segment, circle, internal tangency

Mathematics Subject Classification (2020). 01A27, 51M25

1. INTRODUCTION

Geometry teachers often warn students not to draw inferences from problem figures hold unless the statement of the problem warrants doing so. *Wasanka* were not always so careful. In the presentation of many *wasan* problems, it is assumed that a single circle inscribed in a circular segment is SAGITTAL unless otherwise specified.

The line joining the midpoints of the arc and chord that form a circular segment is called a sagitta. A circle inscribed in a segment passing though these midpoints is called sagittal because the sagitta is one of its diameters. But the line through the center of a segment's arc and the center of *any* circle inscribed in the segment, whether sagittal or not, meets the arc at the point where the inscribed circle touches it internally. This is well illustrated by the proof of the fact that, Figure 1, the diameter of (A) is the harmonic mean of HX and XN. This is Proposition 35 in [3], a well-known *wasan* text; the proof in [3] is short but crucially depends on distinguishing the point where (A) touches XE and the point where AO intersects XE.

Aida Yasuaki poses a more difficult problem based on Figure 1 in the preface to [1]: if the diameters *dai* 'large' 大, *chū* 'medium' 中, and *shō* 'small' 小 in Figure 2 (reproduced from [1]) have lengths $l = 40$, $m = 28$, and $n = 20$, respectively, what are the diameters k and D of the small and large unmarked circles, respectively?

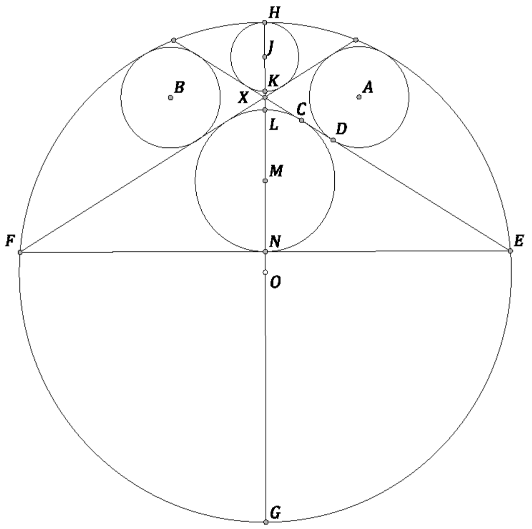

FIGURE 1. The basic setting

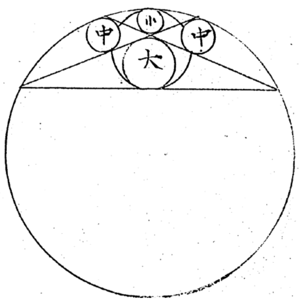

FIGURE 2. The prefatory problem

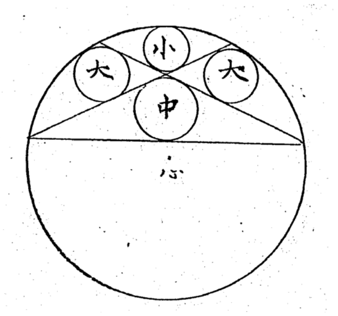

FIGURE 3. The second version of the figure

Without explanation, Aida gives the answers $k = 63$ and $D = 196^{1}/_{3}$, and states (correctly) that these are the results of the formulae $k = \frac{m(l+n)^2}{2ln}$ and $D = \frac{(l/2)^2}{k-(l+n)} + k$.

2. THE MISLEADING FIGURE

Immediately after the preface, Aida's restates this problem but omits the smaller unmarked circle, marks the center of the largest circle as *shin* 'heart' 心, and switches the labels 大 and 中 (Figure 3, also reproduced from [1]). He says nothing about the notational change or the relationship of centers O, A, and B.

Yet auxiliary Figure 4, which appears later in the text clearly shows AO in Figure 1 passing through D as well as the point at which (A) touches (O).

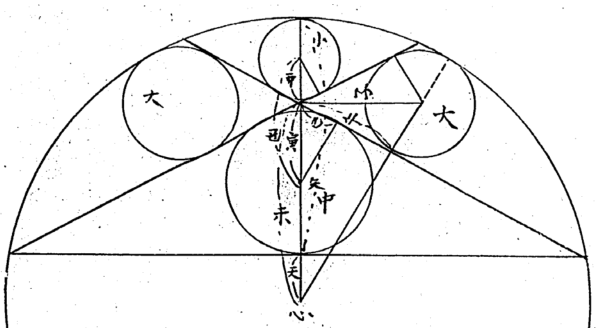

FIGURE 4. The third version of the figure

In other words, (A) is sagittal in Figure 4 even though that is not a given condition in the statement of the problem.

It is easy to construct the incircle of a skewed sector like (A) whether or not the chords involved have equal length [4: 952]. Consider the cyclic quadrilateral $ACDB$ circumscribed by (O) in Figure 5. The incircle (K) of the skewed AEB sector is found by drawing the bisectors of $\angle ABC$ and $\angle AEB$, which meet at incenter I of triangle AEB, and the bisector of $\angle ACB$, which meets arc AB at its midpoint M. The perpendicular to CM through I meets BE at H, and the perpendicular to BE through H meets IE at K. $(K)H$ is the incircle sought.

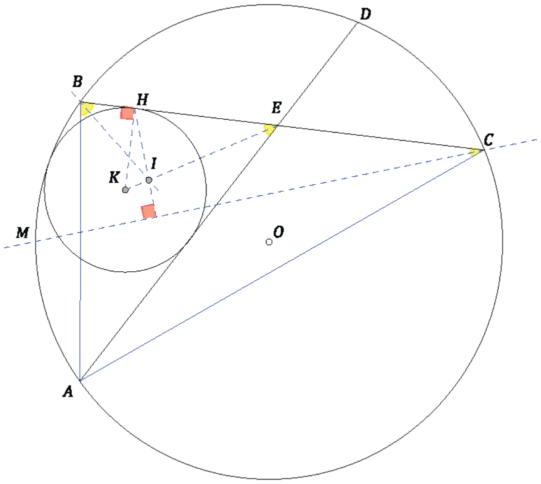

FIGURE 5. Constructing the incircle of a skewed sector

Clearly, the perpendicular to AE through K does not, in general, pass through O even though $AO - KO = KH$, and this is true even in the special case of $AD = BC$. So what are we to make of Aida's solution to the problem if, as Figure 4 suggests, it depends on (A) being sagittal?

3. AIDA'S SOLUTION

What follows is my paraphrase of Aida's solution in modern notation, using radii rather than diameters, as Aida does, and reorganizing his exposition for clarity. We need to define ten lengths: R, p, q, r for the radii of $(O), (M), (J), (A)$ in Figure 1, respectively, and $NE = s, HN = t, NG = u, AX = w, KL = v, XM = x,$ and $JX = y$. Our goal is to find expressions for t and R in terms of p, q, r only.

First, because (M) and (J) are homothetic with respect to X,

$$
\begin{aligned}
\frac{p}{q} &= \frac{x}{y} \\
py - qx &= 0 \\
p(x+y) - px - qx &= 0 \\
p(t-p-q) &= x(p+q) \\
pt - p(p+q) &= x(p+q) \\
\frac{pt}{p+q} &= p + x. \qquad (1)
\end{aligned}
$$

(Aida would, if pressed, have justified the initial proportion in terms of similar triangles, though it would be obviously true to the reader.)

Next, in right triangle XNE, we have

$$
\begin{aligned}
s^2 + (p+x)^2 &= \left[s + \sqrt{x^2 - p^2}\right]^2 \\
s^2 + p^2 + 2px + x^2 &= s^2 + 2s\sqrt{x^2 - p^2} + x^2 - p^2 \\
2px + 2p^2 &= 2s\sqrt{x^2 - p^2} \\
p^2(x+p)^2 &= s^2(x^2 - p^2) \\
p^2(x+p) &= s^2(x-p) \\
p^2\left(\frac{pt}{p+q}\right) &= s^2\left(\frac{pt}{p+q} - 2p\right) \\
p^2\left(\frac{t}{p+q}\right) &= s^2\left(\frac{t}{p+q} - 2\right) \\
p^2 t &= s^2[t - 2(p+q)] \\
p^2 &= uv, \qquad (2)
\end{aligned}
$$

where we have made use of (1). That p is the mean proportional of u and v is, in itself, a rather impressive theorem. Note that (2) implies $p^2 = (2R - t)v = 2Rv - tv$, a fact we will use later.

Now, says Aida,

$$\frac{\sqrt{x^2 - p^2}}{p} = \frac{r}{\sqrt{w^2 - r^2}}$$
$$(x^2 - p^2)(w^2 - r^2) - p^2r^2 = 0.$$

This is follows from the similarity of triangles AXO and ADX. Figure 4 implies this is because A, D, and O are collinear and $AO \parallel CM$, but, as we shall see later, the collinearity is not necessarily the case.

Applying (1) to the first parenthesized factor,

$$x^2 - p^2 = (x + p)(x - p) = \left(\frac{pt}{p + q}\right)\left(\frac{pt}{p + q} - 2p\right) =$$
$$\frac{p^2t^2}{(p + q)^2} - 2p\left(\frac{pt}{p + q}\right) = \frac{p^2t^2 - 2p^2t(p + q)}{(p + q)^2} =$$
$$\frac{p^2t[t - 2(p + q)]}{(p + q)^2} = \frac{p^2tv}{(p + q)^2}.$$

Using (1) together with the fact that AXO is a right triangle, we find an expression for the second factor:

$$w^2 = (R - r)^2 - (R - t + p + x)^2$$
$$w^2 = (R - r)^2 - \left(R - t + \frac{pt}{p + q}\right)^2$$
$$w^2 = (R - r)^2 - \left(R - \frac{qt}{p + q}\right)^2$$
$$w^2 - r^2 = -2Rr + \frac{2Rqt}{p + q} - \frac{q^2t^2}{(p + q)^2}.$$

Thus, $(x^2 - p^2)(w^2 - r^2) - p^2r^2 = 0$ becomes

$$\frac{p^2tv}{(p + q)^2} \cdot \left(-2Rr + \frac{2Rqt}{p + q} - \frac{q^2t^2}{(p + q)^2}\right) - p^2r^2 = 0$$
$$\frac{-2Rrtv}{(p + q)^2} + \frac{2Rqt^2v}{(p + q)^3} - \frac{q^2t^3v}{(p + q)^4} - r^2 = 0$$
$$\frac{-2Rrv(p + q)^2}{t} + 2Rqv(p + q) - q^2tv - \frac{(p + q)^4r^2}{t^2} = 0.$$

Substituting T for $\frac{(p+q)^2r}{t}$, this reduces to $-2RvT + 2Rqv(p+q) - q^2tv - T^2 = 0$, or, substituting S for $2Rv$, $-ST + Sq(p+q) - q^2tv - T^2 = 0$. And because, by (2), we know that $p^2 = 2Rv - tv = S - tv$, we finally arrive at

$$-ST + Sq(p+q) - q^2(S - p^2) - T^2 = 0. \tag{3}$$

The discriminant of the quadratic in T on the left side of (3) turns out to be a perfect square: $S^2 + 4[Sq(p+q) - q^2(S-p^2)] = S^2 + 4[Spq + p^2q^2] = (S+2pq)^2$. Therefore $T = \frac{S \pm (S+2pq)}{-2}$ or, since T as defined must be positive, $T = pq$; hence, $t = \frac{r(p+q)^2}{pq}$. Furthermore, since $2R = \frac{p^2+tv}{v}$, $2R = \frac{p^2}{t-2p-2q} + t$.

It is easy to see that these expressions for t and $2R$ are equivalent to the formulae that Aida gives in his preface when one recalls that we are using radii instead of diameters.

4. CONCLUSION

The clever way Aida solves for T to get at t is highly attractive, but one should not lose sight of the crucial role played by $AXO \sim ADX$. This similarity cannot be proven, as Figure 4 suggests, by assuming that D necessarily lies on AO. Figure 4 is misleading. To remove any ambiguity, let us define D as the point at which (A) touches XE, distinct from the point at which AO crosses XE. $AXO \sim XCM$ because $CM \perp XE$ and $XD \perp XA$; and $XCM \sim XNE$ because $CM \perp XE$ and $XN \perp XE$. I.e., AXO and ADX are similar because each is similar to XCM. (A) and (B) need not be sagittal; D can be anywhere on XE. It is hard to believe that Aida did not observe this.

REFERENCES

[1] Y. Aida, *Sanpō tensei hō* 算法天生法, fascicle 54, 1811, https://yamagata.repo.nii.ac.jp.

[2] N. Nakayama, ed., 2008. *Wasan no zukei kōshiki* (Wasan figures and equations), 2008 (a modern edition of [3] in Japanese and English), http://www.wasan.jp/kosiki/kosiki.html.

[3] K. Yamamoto, H. Hasegawa, *Sanpō jojutsu* 算法助術, 1841, http://www.wasan.jp/archive.html.

[4] P. Yiu, Mixtilinear Incircles, *Am. Math. Monthly*, vol. 106 no. 10 pp. 952–955, 1999.