Sangaku Journal of Mathematics (SJM) ©SJM
ISSN 2534-9562
Volume 10 (2026), pp. 1-8
Received 8 January 2026 Published on-line 14 January 2026.
web: `https://www.sangaku-journal.com/`

Problems involving two circles in a rectangle

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Abstract. We consider a configuration consisting of a rectangle with two circles in it by an algebraic method which sets up several equations and eliminates unnecessary indeterminates from the equations to obtain the requisite relation.

Keywords. Wasan geometry, algebraic method, rectangle with two circles.

Mathematics Subject Classification (2010). 01A27, 51M04

1. Introduction

For a rectangle $ABCD$, let E and F be points on the sides CD and DA, respectively such that the segments BE and CF meet in a point G. The case, the quadrilaterals $ABGF$ and $DEGF$ having incircles, was considered in Wasan geometry (traditional Japanese geometry). In this case the two circles are denoted by α and β, respectively, and we denote the configuration by $\mathcal{R}$ (see Figure 1). In this paper we consider the configuration $\mathcal{R}$ by an algebraic method which sets up several equations from $\mathcal{R}$ and eliminates unnecessary indeterminates from them to obtain the requisite relation. Let a and b be the radii of the circles α and β, respectively and let $h = |AB|$ and $w = |BC|$. We consider using a rectangular coordinate system with origin D so that the point B has coordinates (w, h).

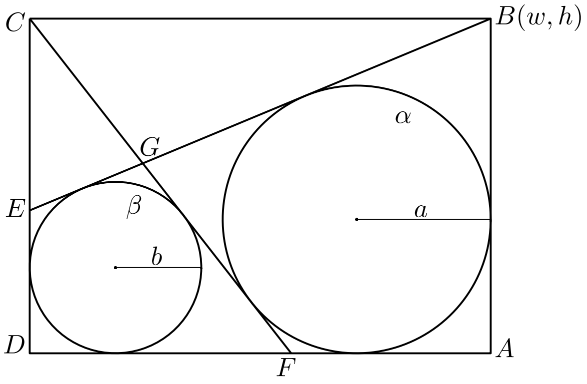

Figure 1: A configuration $\mathcal{R}$.

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2. TWO PROBLEMS

In this section we apply our method to solve two problems in Wasan geometry involving the configuration $\mathcal{R}$. The next problem can be found in [4].

Problem 1. Find b in terms of a and h for the configuration $\mathcal{R}$.

Let (x_a, y_a) and (x_b, y_b) be the coordinates of the centers of the circles α and β, respectively. We can assume that the lines BE and CF are represented by the equations

$$t_1(x,y) = m_1(x-w) - (y-h) = 0 \ \ \textit{and} \ \ t_2(x,y) = m_2x - (y-h) = 0,$$

respectively for real numbers m_1 and m_2. Then we have

$$\frac{|t_1(x_a, y_a)|}{\sqrt{m_1^2+1}} = a.$$

Assume that the perpendicular from the center of α to the side DA meets the line BE in a point of coordinates (x_a, y_a'). There is a real number $z > 0$ such that $y_a = y_a' - z$. Then $t_1(x_a, y_a) = t_1(x_a, y_a' - z) = t_1(x_a, y_a') + z = z > 0$. Therefore we have

$$t_1(x_a, y_a) = a\sqrt{m_1^2+1}. \tag{1}$$

Similarly we have

$$t_2(x_a, y_a) = -a\sqrt{m_2^2+1}, \tag{2}$$

$$t_1(x_b, y_b) = b\sqrt{m_1^2+1} \tag{3}$$

and

$$t_2(x_b, y_b) = b\sqrt{m_2^2+1}. \tag{4}$$

Eliminating m_1, m_2 and w from (1), (2), (3), (4) with $(x_a, y_a) = (w-a, a)$ and $(x_b, y_b) = (b, b)$, we have

$$h\left(bh^2 - (a+b)^2h + 2ab(2a+b)\right) = 4a^2b^2.$$

Solving the equation for b, we have

$$b = \frac{1}{2}h\left(1 \pm \sqrt{\frac{h(h-2a)}{h^2-2ha+4a^2}}\right).$$

However $2b < h$. Therefore we have

$$b = \frac{1}{2}h\left(1 - \sqrt{\frac{h(h-2a)}{h^2-2ha+4a^2}}\right). \tag{5}$$

This is a solution to Problem 1.

The next problem was proposed by Takamori (高盛七郎右衛門) in 1809 ([3]).

Problem 2. Find w in terms of a and h for the configuration $\mathcal{R}$.

Eliminating b, m_1 and m_2 from (1), (2), (3), (4) with $(x_a, y_a) = (w-a, a)$ and $(x_b, y_b) = (b, b)$, we get

$$-4a(h^2 - 2ah + 2a^2)(a-w)^2 + h^3 w(-2a+w) = 0.$$

Solving the equation for w, we have

$$w = a\left(1 \pm h\sqrt{\frac{h}{(h-2a)(h^2-2ha+4a^2)}}\right).$$

However $w > a$. Therefore we have

$$w = a\left(1 + h\sqrt{\frac{h}{(h-2a)(h^2-2ha+4a^2)}}\right). \tag{6}$$

This is a solution to Problem 2.

Fixing h, we can consider w as a function of a with the domain $(0, h/2)$. Let $f(a) = (h-2a)(h^2-2ha+4a^2)$. Then $f(a) > 0$ and

$$\frac{df(a)}{da} = -4((h-2a)^2 + 2a^2) < 0.$$

Hence $f(a)$ is a strongly decreasing function of a. Since $w = a\left(1 + h\sqrt{h/f(a)}\right)$, *$w$ is a strongly increasing function of a.* Therefore *the shape of the rectangle $ABCD$ is uniquely determined by the ratio $h : a$.*

3. The incircle of the triangle CEG

Let δ be the incircle of the triangle CEG with radius d. In this section we consider the following problem proposed by Hachiya[2] (蜂谷吉治) in 1875 ([5]) (see Figure 2).

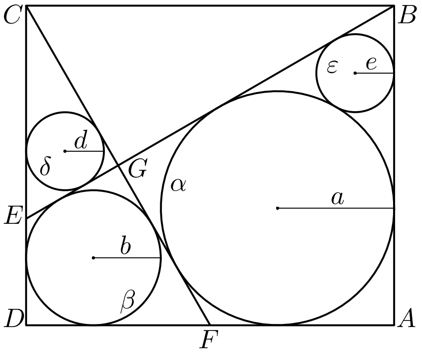

Figure 2.

Problem 3. Let ε be the circle with radius e touching the segments AB and BG and the circle α externally for the configuration $\mathcal{R}$. If $d = e$, find a in terms of d.

Hachiya gave the answer $a = 3d$. However we show that there is one more answer. Let (x_d, y_d) be the coordinates of the center of the circle δ. We have

$$t_1(x_d, y_d) = -d\sqrt{m_1^2 + 1} \tag{7}$$

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[2] The surname can also be read “Hachitani”.

and

$$t_2(x_d, y_d) = d\sqrt{m_2^2 + 1}. \tag{8}$$

Eliminating b, m_1, m_2, w and y_d from (1), (2), (3), (4), (7) and (8) with $(x_a, y_a) = (w - a, a)$, $(x_b, y_b) = (b, b)$ and $x_d = d$ we have

$$(h-a)^2(h^2 - 2ah + 4a^2)d^2 - a^4h(h - 2a) = 0.$$

Solving the equation for d, we have

$$d = \frac{a^2}{h-a}\sqrt{\frac{h(h-2a)}{h^2 - 2ah + 4a^2}}. \tag{9}$$

Let (x_e, y_e) be the coordinates of the center of the circle ε. We have

$$t_1(x_e, y_e) = e\sqrt{m_1^2 + 1}. \tag{10}$$

Eliminating b, m_1, m_2 and w from (1), (2), (3), (4) and (10) with $(x_a, y_a) = (w - a, a)$, $(x_b, y_b) = (b, b)$ and $(x_e, y_e) = (w - e, a + 2\sqrt{ae})$, we have

$$(h - a)e + 2a\sqrt{ae} - a(h - a) = 0.$$

Therefore we have

$$e = a\left(\frac{-a + \sqrt{h^2 - 2ah + 2a^2}}{h-a}\right)^2. \tag{11}$$

Solving the equation $d = e$ with (9) and (11) for a, we have two positive real solutions:

$$a = \frac{-1+\sqrt{3}}{2}h \quad \textit{and} \quad a = 0.47885\cdots h.$$

If $a = \left(-1 + \sqrt{3}\right)h/2$, then $w = 2h/\sqrt{3}$ by (6), and $b = \sqrt{3}(-1+\sqrt{3})h/6$ and $d = (-1 + \sqrt{3})h/6$ by (5) and (9), respectively. Hence $b = \sqrt{3}d$ and $a = 3d$. The last equation is the answer given by Hachiya. Figure 2 shows the configuration $\mathcal{R}$ in this case.

If $a = 0.47885\cdots h$, then $w = 2.85630\cdots h$, and $b = 0.39504\cdots h$ and $d = 0.092365\cdots h$. Hence $b = 4.27692\cdots d$ and $a = 5.18438\cdots d$. The last equation is the answer Hachiya did not mention. Figure 3 shows $\mathcal{R}$ in this case.

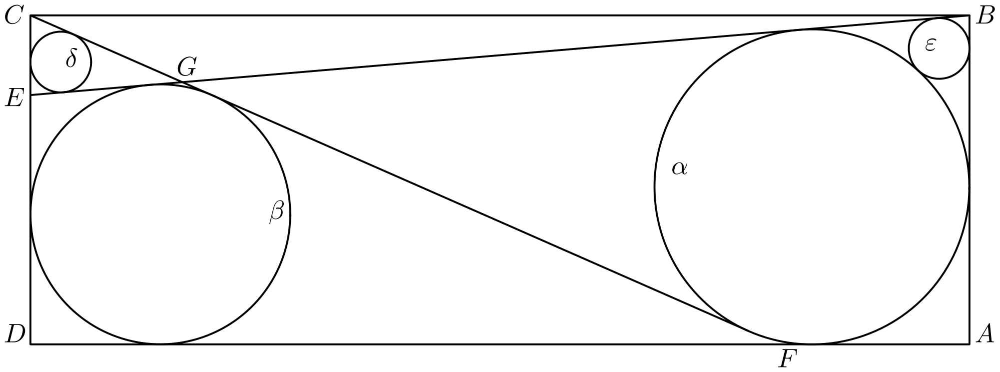

Figure 3.

4. THE INCIRCLE OF THE TRIANGLE BCG

In this section we consider the incircle of the triangle BCG. Let γ be the incircle of BCG with radius c. Let (x_c, y_c) be the coordinates of the center of γ. We have

$$t_1(x_c, y_c) = -c\sqrt{m_1^2+1} \tag{12}$$

and

$$t_2(x_c, y_c) = -c\sqrt{m_2^2+1}. \tag{13}$$

Eliminating b, m_1, m_2, w and x_c from (1), (2), (3), (4), (12) (13) with $(x_a, y_a) = (w-a, a)$, $(x_b, y_b) = (b, b)$ and $y_c = h - c$, we get

$$-4a^2c^2 + h(h-2a)(a^2-c^2) = 0.$$

Therefore we have

$$c = a\sqrt{\frac{h(h-2a)}{h^2-2ah+4a^2}}. \tag{14}$$

Onodera (小野寺倉吉) proposed the next problem with the answer $a = 3d$ in 1836 ([1]) (see Figure 4):

Problem 4. If $b = c$, find a in terms of d for the configuration $\mathcal{R}$.

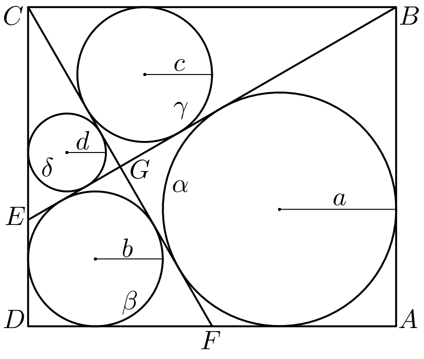

Figure 4.

Solving $b = c$ with (5) and (14), we get $a = (-1+\sqrt{3})h/2$. Since the shape of the rectangle $ABCD$ is uniquely determined by the ratio $h : a$, the rectangles considered by Hachiya and this problem are similar. Hence we get Onodera's answer $a = 3d$. A simple geometric solution to this problem can be found in [2].

Using (14), (6) can be rearranged as

$$w = a + \frac{hc}{h-2a}.$$

Also (5) and (9) can be written as

$$b = \frac{h(a-c)}{2a} \quad \textit{and} \quad d = \frac{ac}{h-a}.$$

Eliminating h from the last two equations we get a simple relation of the radii of the four circles:

$$2bd + (c-a)(c+d) = 0.$$

5. Some notable segments of $\mathcal{R}$

In this section we show that there are several segments whose length equal the radii of the circles α and γ for the configuration $\mathcal{R}$. Assume that the circle γ touches the side BC at a point H (see Figure 5). We have the next theorem.

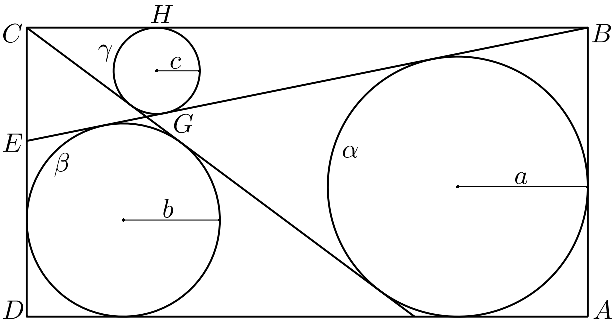

Figure 5.

Theorem 1. *We have $|CH| = a$ for the configuration $\mathcal{R}$.*

Proof. Eliminating c, m_1, m_2, h, w from (1), (2), (3), (4), (12), and (13) with $(x_a, y_a) = (w-a, a)$, $(x_b, y_b) = (b, b)$ and $y_c = h - c$, we have $a - x_c = 0$. Therefore we get $|CH| = x_c = a$. $\square$

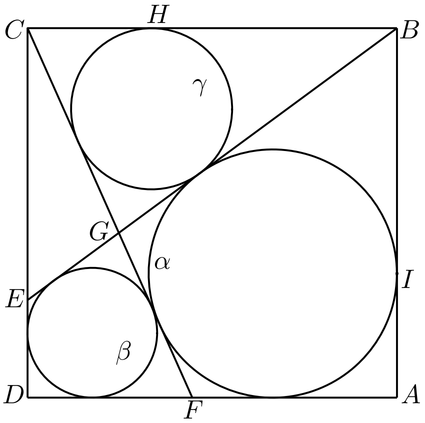

Figure 6.

Corollary 1. *The rectangle $ABCD$ is a square if and only if the circles α and γ touch for the configuration $\mathcal{R}$.*

Proof. Assume that the circle α touches the side AB at a point I (see Figure 6). Then $|BH| = |BI|$ if and only if the circles α and γ touch. While $|CH| = |AI|$ by Theorem 1. Therefore the corollary is proved. $\square$

We consider segments with length c for the configuration $\mathcal{R}$. We use the next known lemma (see Figure 7).

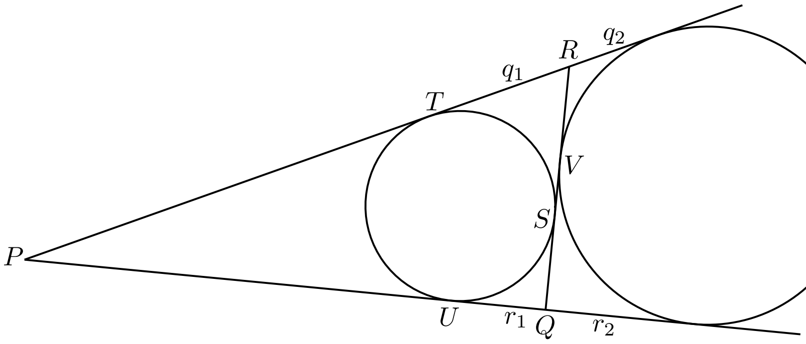

Figure 7.

Lemma 1. *For a triangle PQR, the incircle touches the sides QR, RP and PQ at points S, T and U, respectively. The excircle of PQR touching QR from the side opposite to P touches QR at a point V. Then we have $|QS| = |RV|$ and $|QV| = |RS|$.*

Proof. Let $|QS| = |QU| = r_1$, $|RS| = |RT| = q_1$, $|QV| = r_2$ and $|RV| = q_2$. Then we have $r_1 + q_1 = r_2 + q_2$ and $r_1 + r_2 = q_1 + q_2$. Adding both sides we have $2r_1 + q_1 + r_2 = q_1 + r_2 + 2q_2$, i.e., $r_1 = q_2$. Then we also have $r_2 = q_1$. ☐

The point of tangency of α (resp. β, δ) and the segment FA (resp. FG, CE) is denoted by J (resp. K, L) (see Figure 8).

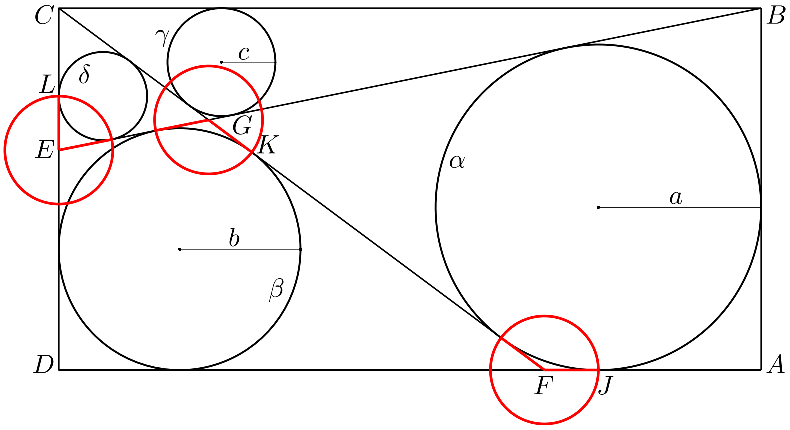

Figure 8.

Theorem 2. *We have $|FJ| = |GK| = |EL| = c$ for the configuration $\mathcal{R}$.*

Proof. Let x_f be the x-coordinate of the point F. Then $t_2(x_f, 0) = 0$, i.e., $x_f = -h/m_2$. Therefore $|FJ| = w - a + h/m_2$. Let

$$d_f = w - a + \frac{h}{m_2} - c. \tag{15}$$

Eliminating a, c, m_1, m_2, h and w from (1), (2), (3), (4), (12), (13) and (15) with $(x_a, y_a) = (w - a, a)$, $(x_b, y_b) = (b, b)$ and $(x_c, y_c) = (a, h - c)$ by Theorem 1, we get $d_f = 0$. Hence $|FJ| = c$. The rest of the theorem is proved by Lemma 1. ☐

In Figure 8 the circle with center E (resp. F, G) passing through the point L (resp. J, K) is congruent to the circle γ and denoted in red.

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