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Haga's theorems in paper folding and related theorems in Wasan geometry Part 1

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Abstract. Haga's fold in paper folding is generalized. Recent generalization of Haga's theorems and problems in Wasan geometry involving Haga's fold are also generalized.

Keywords. Haga's fold, Haga's theorems, incircles and excircles of right triangles

Mathematics Subject Classification (2010). 01A27, 51M04

1. Introduction

Let $ABCD$ be a piece of square paper with a point E on the side DA. We fold the paper so that the corner C coincides with E and the side BC is carried into GE, which meets the side AB in a point F (see Figure 1). We call this Haga's fold. Unifying Haga's theorems in paper folding in [2], we obtained the following theorem [8].

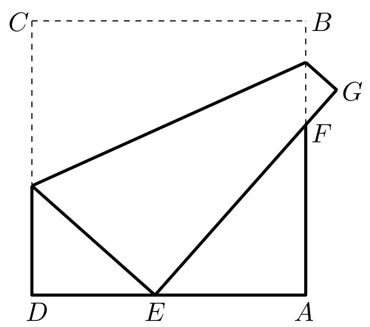

Figure 1.

Theorem 1.1. *The relation $|AE||AF| = 2|DE||BF|$ holds for Haga's fold.*

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In this paper we generalize Haga's fold and show that Theorem 1.1 holds for the generalized Haga's fold. Haga's fold was also considered in Wasan geometry, and one of the most famous results says that the inradius of the right triangle AFE equals the overhang $|FG|$. We also generalize this result and show that not only the incircle of AFE, but the excircles of it play important roles. There are two triangles similar to AFE in the figure made by the generalized Haga's fold. We show that there are several simple relationships between the incircles and the excircles of the three similar triangles.

2. Preliminaries

In this section we summarize results for incircles and excircles of triangles, which are used in later sections. Let $a = |BC|$, $b = |CA|$ and $c = |AB|$ for a triangle ABC.

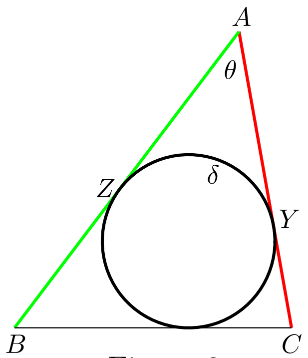

Figure 2.

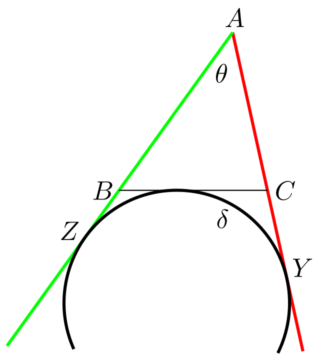

Figure 3.

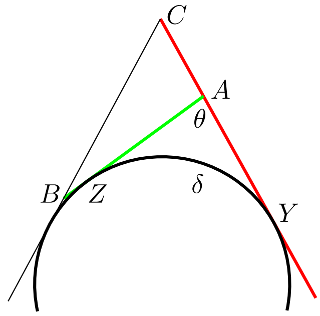

Figure 4.

Theorem 2.1. *Let δ be the incircle or one of the excircles of a triangle ABC touching the lines AB and CA at points Z and Y, respectively. If θ is the angle subtended by δ from A, we get*

$$\sin^2 \frac{\theta}{2} = \frac{|BZ||CY|}{|AB||AC|}.$$

Proof. There are two cases to be considered: (i) δ is the incircle of ABC or δ touches BC from the side opposite to A, i.e., $\theta = \angle CAB$ (see Figures 2 and 3). (ii) δ touches the side AB (resp. CA) from the side opposite to C (resp. B), i.e., $\theta + \angle CAB = 180°$ (see Figure 4). We prove the case (ii). Let us assume that δ touches the side AB from the side opposite to C. Then

$$\begin{aligned}\sin^2 \frac{\theta}{2} &= \frac{1-\cos\theta}{2} = \frac{1+\cos\angle CAB}{2} = \frac{1+(b^2+c^2-a^2)/(2bc)}{2} \\ &= \frac{(b+c)^2-a^2}{4bc} = \frac{(a+b+c)(b+c-a)}{4bc} = \frac{|BZ||CY|}{bc},\end{aligned}$$

because $a+b+c = a+b+|BZ|+|AY| = (a+|BZ|)+(b+|AY|) = 2|CY|$, and $b+c-a = (|CY|-|AZ|)+(|AZ|+|BZ|)-(|CY|-|BZ|) = 2|BZ|$. The rest of the theorem can be proved in a similar way. □

Let γ, γ_a be the incircle of ABC, the excircle of ABC touching BC from the side opposite to A, respectively. Let r, r_a be the radii of γ, γ_a, respectively.

Theorem 2.2. *If ABC is a right triangle with right angle at A, the following statements hold.*

(i) *The remaining common tangent of each of the pairs* γ_a *and* γ_b, γ_a *and* γ_c, γ *and* γ_b, γ *and* γ_c *is perpendicular to* BC [9], *and* $r_a = r + r_b + r_c$ *holds* [3].
(ii) $rr_a = r_b r_c$ [3].
(iii) $r + r_b = b$ *and* $r + r_c = c$.

Proof. $r + r_b = (-a + b + c)/2 + (a + b - c)/2 = b$. This proves (iii). □

3. Generalized Haga's fold

In this section, we generalize Haga's fold and Theorem 1.1. The figure made by Haga's fold is uniquely determined if we fix a point E on the side DA for a square $ABCD$. We now consider E is a point on the *line* DA for the square $ABCD$ and assume that m is the perpendicular bisector of the segment CE, G is the reflection of B in m, and F is the point of intersection of the lines AB and EG if they intersect, where we define $F = B$ if $E = A$. The figure consisting of $ABCD$ and the points E, F (if exists) and G is denoted by $\mathcal{H}$. If $ABCD$ is a piece of square paper and m passes through the inside of $ABCD$, we can really fold the paper with the crease line m so that BC is carried into GE (see Figures from 7 to 11). We thus call $\mathcal{H}$ the figure made by generalized Haga's fold, and m the crease line of $\mathcal{H}$.

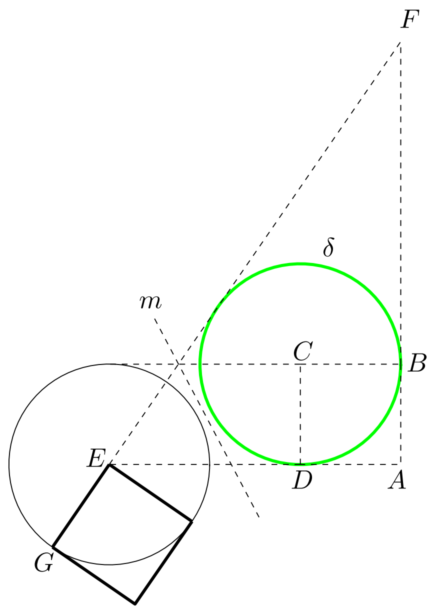

Figure 5: (h1)

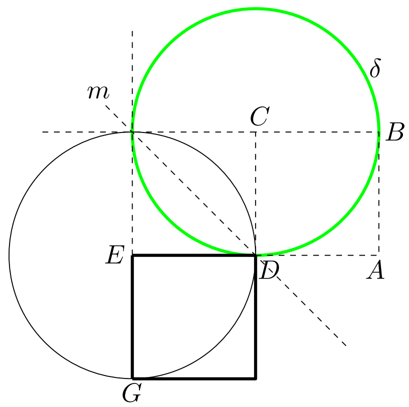

Figure 6: (h2)

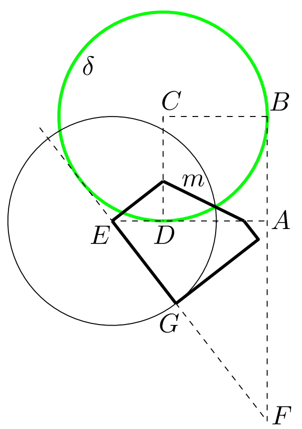

Figure 7: (h3)

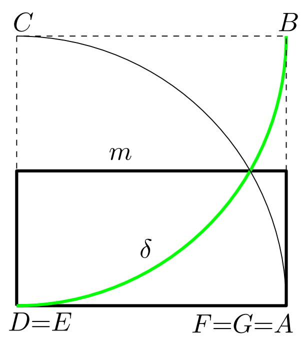

Figure 8: (h4)

Let $b = |DE|$, $d = |AB|$. There are seven cases to be considered for $\mathcal{H}$:
(h1) D lies between E and A and $b > d$ (see Figure 5),
(h2) D is the midpoint of EA, i.e., $b = d$ (see Figure 6),
(h3) D lies between E and A and $b < d$ (see Figure 7),
(h4) $D = E$ (see Figure 8),
(h5) E lies between D and A (see Figure 9),
(h6) $E = A$ (see Figure 10),
(h7) A lies between D and E (see Figures 11, 12, 13).

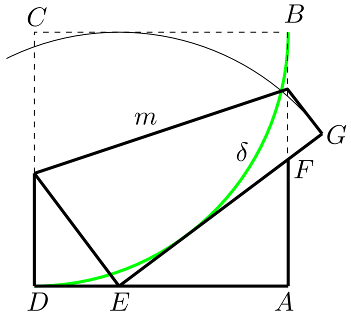

Figure 9: (h5)

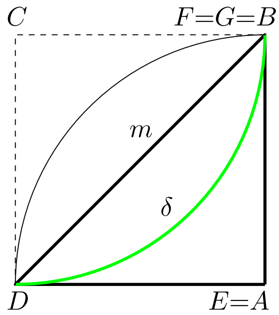

Figure 10: (h6)

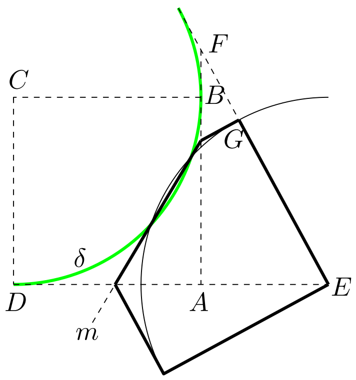

Figure 11: (h7), $|AE| < \sqrt{2}d$

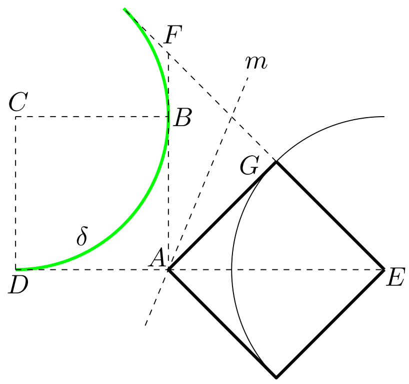

Figure 12: (h7), $|AE| = \sqrt{2}d$

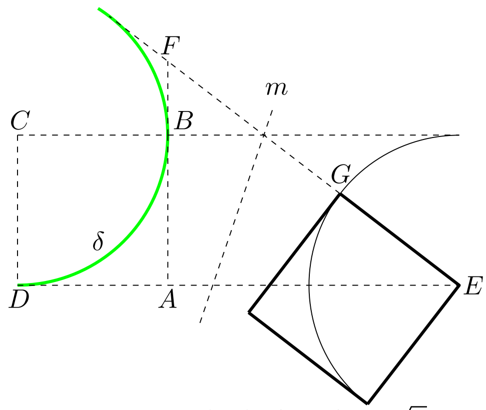

Figure 13: (h7), $|AE| > \sqrt{2}d$

The point F does not exists only in the case (h2). Let $c = |BF|$, if F exist. Let δ be the circle with center C passing through B.

Proposition 3.1. *The followings hold for $\mathcal{H}$.*
(i) *The line EG touches the circle δ.*

(ii) $|EF| = \begin{cases} c-b & \textit{in the case } \text{(h3)}, \\ b+c & \textit{in the cases } \text{(h1), (h4), (h5), (h6)}, \\ b-c & \textit{in the case } \text{(h7)}. \end{cases}$

Proof. The circle with center E passing through G has radius d, and touches the line BC (see Figures 5 to 13). While the reflections of this circle and the line BC in the crease line m are δ and the line EG. This proves (i). The part (ii) follows from (i). □

Notice that $b = 0$ and $|AF| = 0$ are equivalent, also $c = 0$ and $|AE| = 0$ are equivalent. Therefore $|AE||AF| = 0$ if and only if $bc = 0$. The next theorem is a generalization of Theorem 1.1.

Theorem 3.1. *If the lines AB and EG intersect, $|AE||AF| = 2bc$ holds for $\mathcal{H}$.*

Proof. If $|AE||AF| = 0$, $bc = 0$. If $|AE||AF| \neq 0$, we consider the points E, F as B, C in Theorem 2.1 and D, B as Z, Y, respectively, and get

$$\frac{bc}{|AE||AF|} = \sin^2 45^\circ = \frac{1}{2}.$$

□

4. INCIRCLE AND EXCIRCLES OF THE TRIANGLE AFE

From now on we assume that the lines AB and EG intersect. Therefore the case (h2) is not considered. We consider the result in Wasan geometry saying that the inradius of AFE equals the overhang $|FG|$ in the case (h5) [1], [5], [10], [11]. The result has been quoted from the sangaku problem dated 1893 in [1], but the book [10] is older. Several new results related to this are also given in this section. We now call (h1), (h3), (h5) and (h7) the ordinary cases, and (h4) and (h6) the degenerate cases.

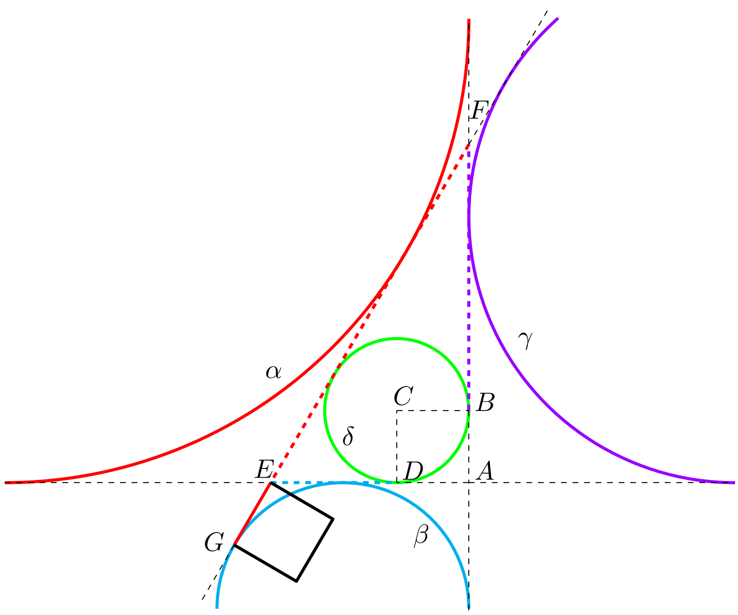

Figure 14: (h1)

We define three circles (see Figures 14 to 19). Let $\alpha(\neq \delta)$ be the incircle or one of the excircles of AFE with center lying on the line AC in the ordinary cases, the point A in the degenerate cases. Let $\beta(\neq \alpha)$ be the incircle or one of the excircles of AFE with center lying on the line joining E and the center of α in the ordinary cases, the point A in the case (h4), and the reflection of δ in the line AB in the case (h6). Let $\gamma(\neq \alpha)$ be the incircle or one of the excircles of AFE with center lying on the line joining F and the center of α in the ordinary cases, the reflection of δ in the line DA in the case (h4), and the point A in the case (h6). Let $a = |FG|$.

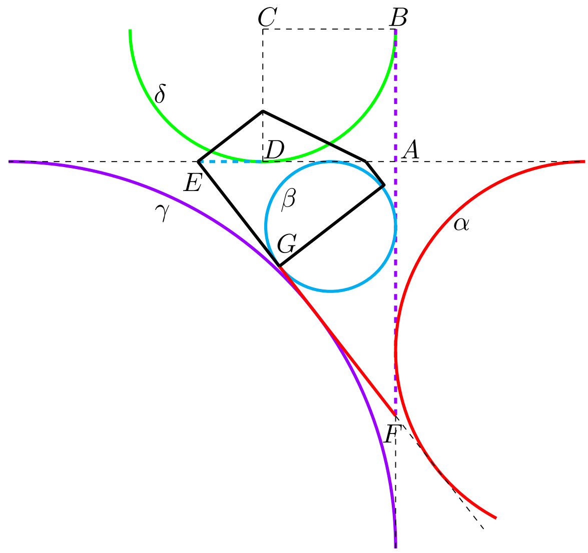

Figure 15: (h3)

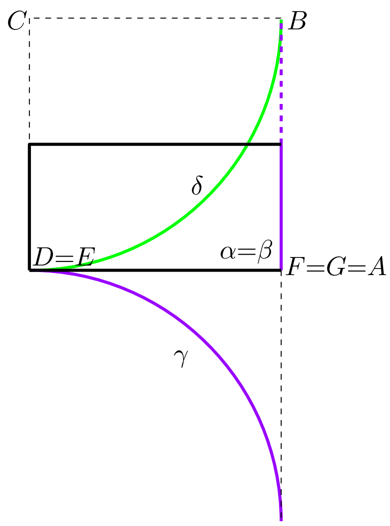

Figure 16: (h4)

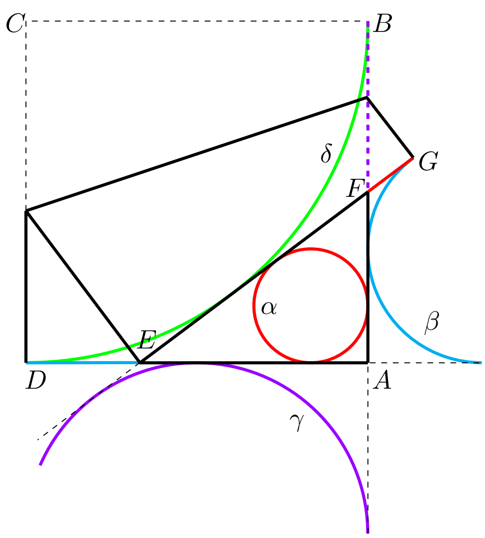

Figure 17: (h5)

Theorem 4.1. *The radii of the circles α, β and γ equal a, b and c, respectively.*

Proof. We prove that the radius of α equals a. The degenerate cases are trivial. The proof in the ordinary cases is similar to that for Haga's fold given in [4]. We prove the case (h1) (see Figure 14). By Proposition 3.1(ii), the radius of α equals

$$\frac{|AE|+|AF|+|EF|}{2}=\frac{(b+d)+(c+d)+|EF|}{2}=d+|EF|=a.$$

The other cases can be proved in a similar way. We now prove that the radius of β equals b. The degenerate cases are obvious. In the case (h1) (see Figure 14), the radius of β equals

$$\frac{|AE|-|AF|+|EF|}{2}=\frac{(b+d)-(c+d)+(b+c)}{2}=b$$

by Proposition 3.1(ii). The rest of the theorem can be proved in a similar way. □

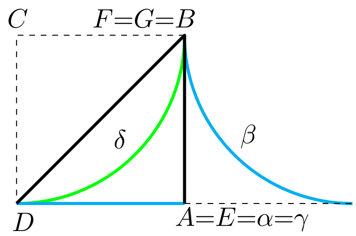

Figure 18: (h6)

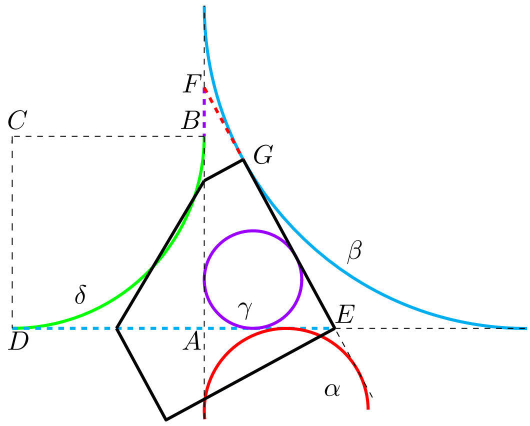

Figure 19: (h7)

The set $\{\alpha,\beta,\gamma,\delta\}$ consists of the incircle and the excircles of the triangle AFE, if AFE is proper (see Figures 20 to 23). Therefore $a=b+c+d$, $c=a+b+d$, $d=a+b+c$, $b=a+c+d$ hold in the cases (h1), (h3), (h5), (h7), respectively by Theorem 2.2(i). The equation $d=a+b+c$ also holds in the degenerate cases. Hence we get (i) of the next theorem. The part (ii) in the ordinary cases follows from Theorem 2.2(ii) and the degenerate cases are trivial.

Theorem 4.2. *The following relations hold.*

(i) $-a+b+c+d=0$ *in the case* (h1),
$a+b-c+d=0$ *in the case* (h3),
$a+b+c-d=0$ *in the case* (h4), (h5), (h6),
$a-b+c+d=0$ *in the case* (h7).

(ii) $ad=bc$.

Theorem 4.3. *The circle β touches EG at the point G or coincides with G.*

Proof. The degenerate cases are trivial. In the cases (h1) and (h3), the distance between E and the point of contact of β and the line DA equals $|AE|-b=(b+d)-b=d$, i.e., β touches EG at G. The other cases can be proved similarly. □

Recall that the figure $\mathcal{H}$ is uniquely determined if a point E on the line DA is fixed for a square $ABCD$. For a triangle $A'F'E'$ with right angle at A', we can construct $\mathcal{H}$ such that $A=A'$, $F=F'$, $E=E'$ as follows: Let $A=A'$, $F=F'$, $E=E'$ and let δ be the incircle or one of the excircles of AFE with center C. Let B and D be the feet of perpendiculars from C to the lines AF and AE, respectively. Then the square $ABCD$ with the point E form a desired figure $\mathcal{H}$.

Since there are four choices of the circle δ for AFE, we can construct four such figures.

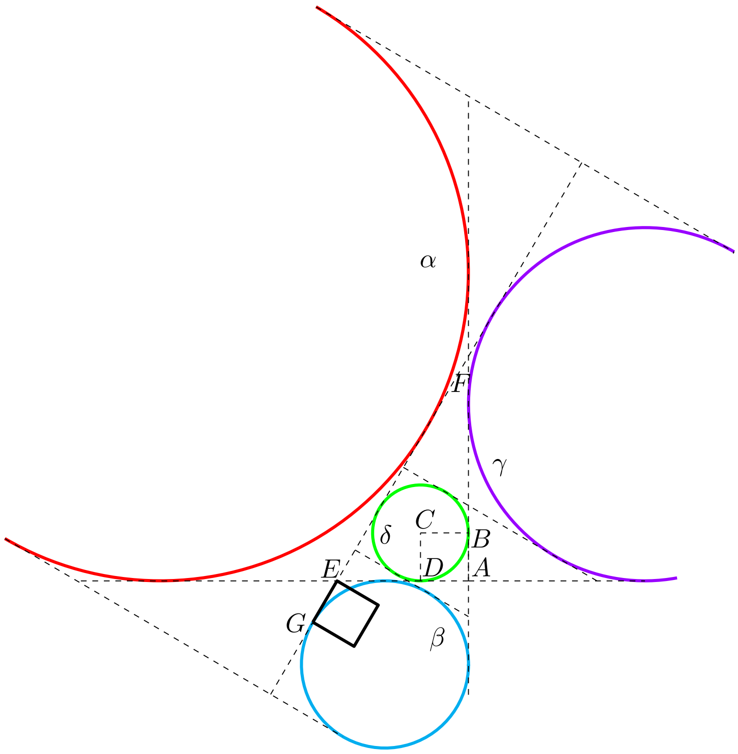

Figure 20: $a = b + c + d$ in the case (h1)

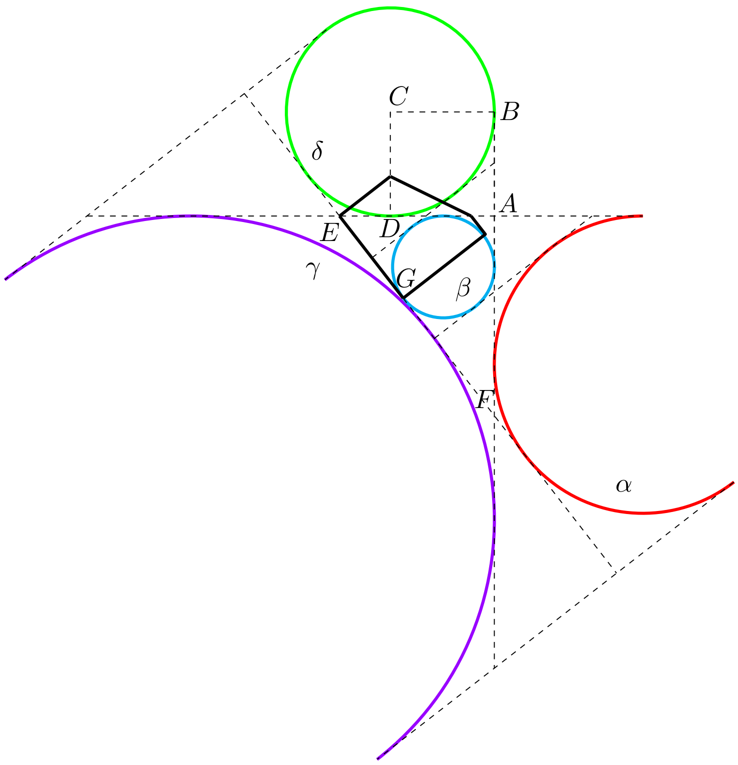

Figure 21: $c = a + b + d$ in the case (h3)

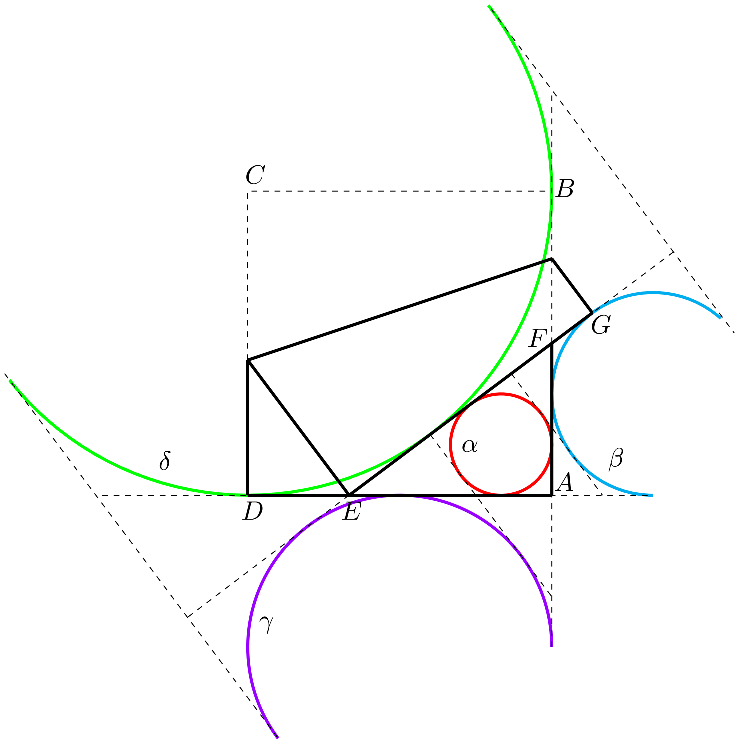

Figure 22: $d = a + b + c$ in the case (h5)

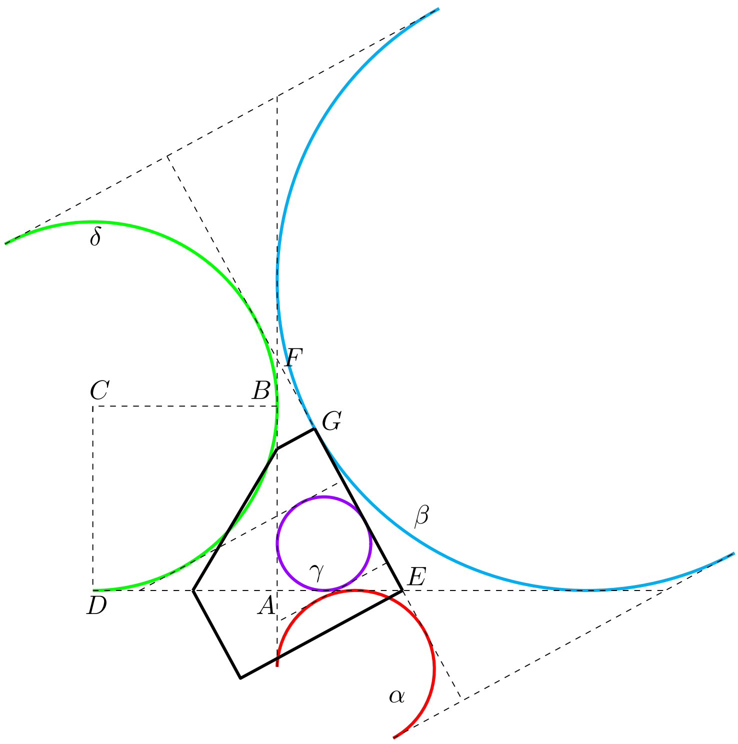

Figure 23: $b = a + c + d$ in the case (h7)

5. Circumcircle of AFE

The circumcircle of AFE and the incircle of the triangle made by the lines BC, CD and EG are congruent in the case (h5) (see Figure 26) [7]. In this section we generalize this fact. Notice that m and AC are parallel only in the case (h2) and intersect in the other cases. Let ε_1 be the circle with center at the point of intersection of m and AC touching the line BC (see Figures 24 to 27). The definition shows that ε_1 touches EG and CD, and coincides with the incircle of $ABCD$ in the degenerate cases. Let r_1 be the radius of ε_1. The next theorem shows that if AFE is a proper triangle, ε_1 and the circumcircle of AFE are congruent and one of the two circles passes through the center of the other.

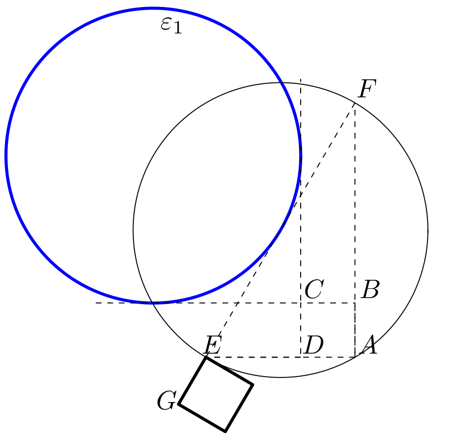

Figure 24: (h1)

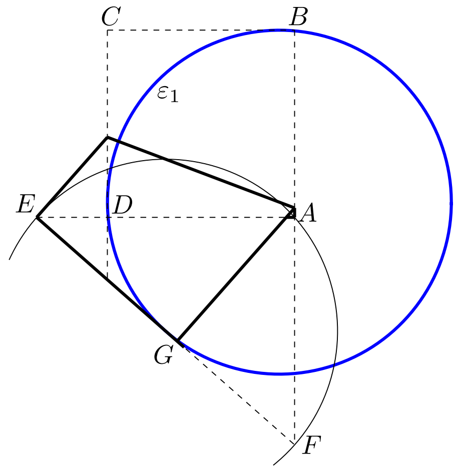

Figure 25: (h3)

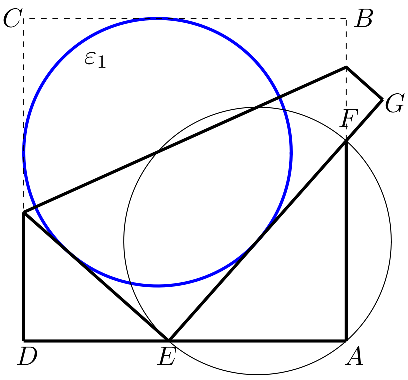

Figure 26: (h5)

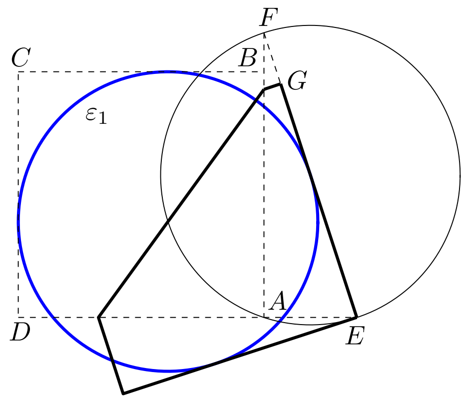

Figure 27: (h7)

Theorem 5.1. *The following statements are true.*
(i) $r_1 = |EF|/2$. (ii) ε_1 *touches* EF *at its midpoint.*

Proof. We prove (i). The degenerate cases are obvious. Assume that the lines BC and EG meet in a point K and $k = |BK|$. We prove the case (h3) (see Figure 28). Since the triangles BFK and AFE are similar, $k/|BF| = |AE|/|AF|$, i.e., $k = c|AE|/|AF|$. Therefore by Theorem 4.2(ii),

$$|CK| = k - d = \frac{c|AE| - d|AF|}{|AF|} = \frac{c(b+d) - d(c-d)}{|AF|} = \frac{bc + d^2}{|AF|} = \frac{d(a+d)}{|AF|}.$$

From the similar triangles CGK and AFE, we have $|CK|/r_1 = |AE|/a$. Therefore by Theorems 3.1, 4.2 and Proposition 3.1(ii), we get

$$r_1 = \frac{|CK|}{|AE|}a = \frac{d(a+d)a}{|AE||AF|} = \frac{ad(c-b)}{2bc} = \frac{c-b}{2} = \frac{|EF|}{2}.$$

This proves (i). The other case can be proved similarly. We prove (ii). The degenerate cases are trivial. The perpendicular to EF at E touches ε_1, for it is the reflection of the line CD in m and CD touches ε_1. Therefore the perpendicular to EF at F also touches ε_1 by (i). This proves (ii). □

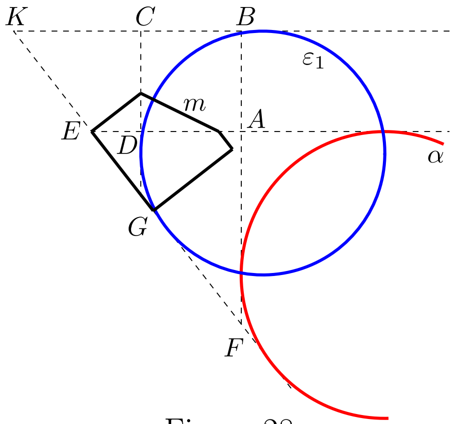

Figure 28.

6. Incircles and excircles of another triangles

Let us assume that the crease line m meets the lines AB and CD in points H and I, respectively for $\mathcal{H}$. In this section we consider the incircles and the excircles of the triangles GFH and DEI, which are similar to AFE. The sum of the inradii of the triangles GFH and DEI equals a in the case (h5) [6], [10] (see Figure 29). We generalize this fact and give several new results.

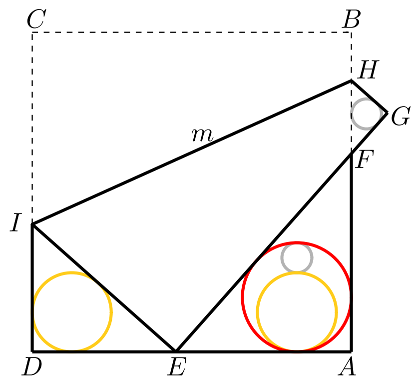

Figure 29.

Proposition 6.1. *The following relation holds for* $\mathcal{H}$.

$$a = \begin{cases} |FH| - |DI| & \text{in the cases (h1), (h3),} \\ |DI| - |FH| & \text{in the cases (h4), (h5), (h6), (h7).} \end{cases}$$

Proof. It is sufficient to consider the ordinary cases. We prove the cases (h1) and (h3) (see Figures 30 and 31). Let T be the point of contact of the circle δ and the line EG. The lines DT and m are parallel, since DT is perpendicular to CE. Let J be the point of intersection of the lines AB and DT. Then the triangles DAJ and CTE are congruent. Hence $|AJ| = |ET|$. Therefore we get $|GT| = |BJ|$, while T is the reflection of B in the line CF. Therefore J is the reflection of G in CF, i.e., $|FJ| = a$. Hence $|DI| = |HJ| = |FH| - |FJ| = |FH| - a$. The other cases can be proved similarly. □

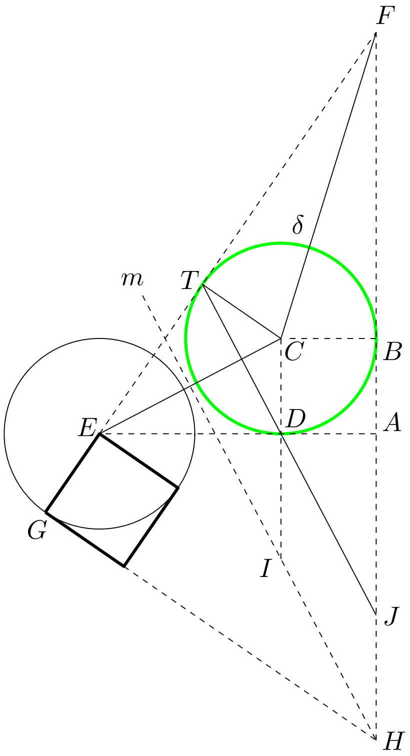

Figure 30: (h1)

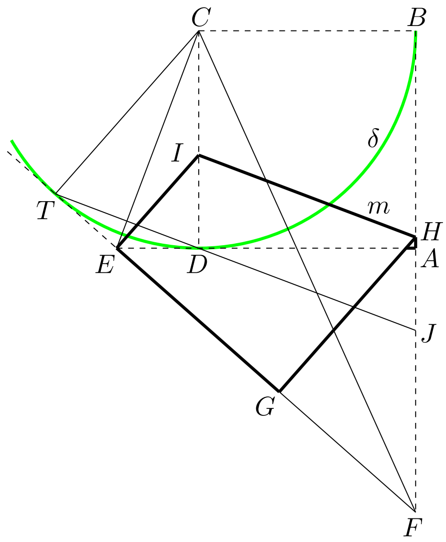

Figure 31: (h3)

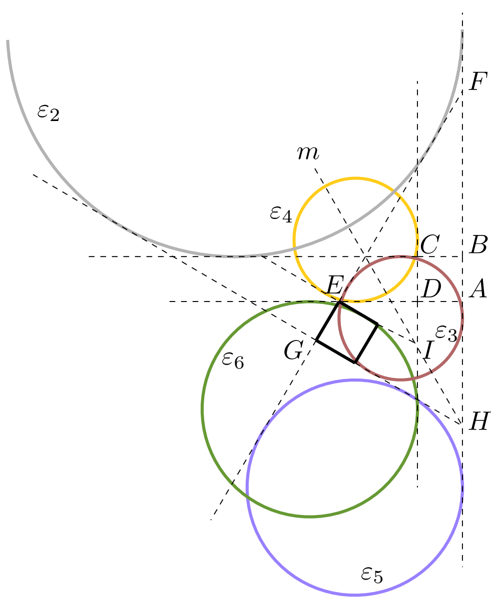

Figure 32: (h1)

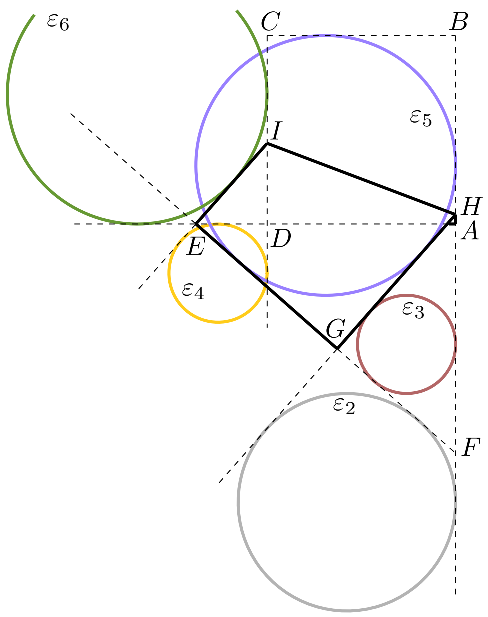

Figure 33: (h3)

We define three circles (see Figures 32 to 35). Let ε_2 be the excircle of GFH touching FG from the side opposite to H in the cases (h1) and (h3), the point A (resp. B) in the case (h4) (resp. (h6)), and the incircle of GFH in the cases (h5) and (h7). Let ε_3 be the incircle or one of excircles of GFH touching FG from the side opposite to ε_2 in the ordinary cases, and the point A (resp. B) in the case (h4) (resp. (h6)). Let ε_4 be the excircle of DEI touching DE from the side opposite to I in the cases (h1) and (h3), the point D in the degenerate cases, and the incircle of DEI in the cases (h5) and (h7). Let r_i $(i = 2, 3, 4)$ be the radius of ε_i.

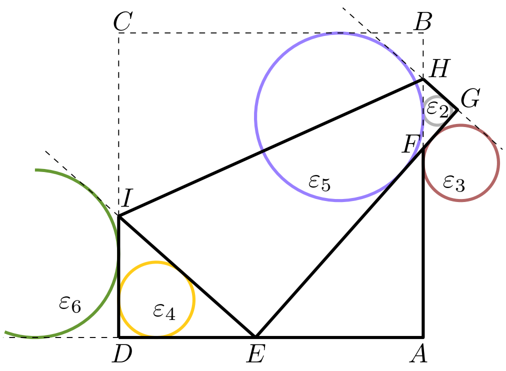

Figure 34: (h5)

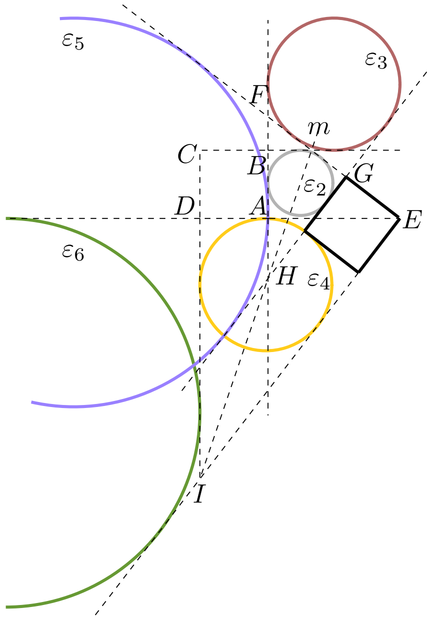

Figure 35: (h7)

Theorem 6.1. *The following relations are true for* $\mathcal{H}$.
(i) $r_3 = r_4$. (ii) $a = r_2 + r_4$.

Proof. We prove (i). It is sufficient to prove the ordinary cases. In the case (h1), we have

$$r_3 = \frac{|FG| + |GH| - |FH|}{2} = \frac{a - (|FH| - |BH|)}{2} = \frac{a - c}{2}$$

and

$$r_4 = \frac{|DE| - |DI| + |EI|}{2} = \frac{b - |DI| + d + |DI|}{2} = \frac{b + d}{2}.$$

Therefore $r_3 = r_4$ by Theorem 4.2(i). In the case (h3), we have

$$r_3 = \frac{|FG| + |GH| - |FH|}{2} = \frac{a + c - 2|FH|}{2}$$

and

$$r_4 = \frac{|DE| - |DI| + |EI|}{2} = \frac{b - 2|DI| + d}{2}.$$

Therefore by Theorem 4.2(i) and Proposition 6.1,

$$r_3 - r_4 = \frac{a + c - b - d - 2|FH| + 2|DI|}{2} = a - |FH| + |DI| = 0.$$

The rest of (i) can be proved similarly. By Theorem 2.2(iii), $a = r_2 + r_3 = r_2 + r_4$. This proves (ii). □

We consider two more circles (see Figures 32 to 35). Let $\varepsilon_5 (\neq \varepsilon_3)$ be the incircle or one of the excircles of GFH with center lying on the line joining F and the center of ε_3 in the ordinary cases, the incircle of $ABCD$ in the case (h4), and the point B in the case (h6). Let $\varepsilon_6 (\neq \varepsilon_4)$ be the incircle or one of the excircles of DEI with center lying on the line joining E and the center of ε_4 in the ordinary cases, the reflection of the incircle of $ABCD$ in the line CD in the case (h4), and the point D in the case (h6). The proof of the next theorem is similar to that of Theorem 6.1(i) and is omitted.

Theorem 6.2. *The circles ε_5 and ε_6 are congruent.*

If t is the remaining common tangent of ε_3 and ε_5 in the ordinary cases, then t is perpendicular to AB by Theorem 2.2(i). Hence the triangle formed by the lines GH, AB and t is congruent to DEI. Therefore the figure consisting of ε_3 and ε_5 is congruent to the figure consisting of ε_4 and ε_6. The statement is also true in the degenerated cases.

7. Special case

The crease line m touch the circle δ if and only if $|DE| = \sqrt{3}d$. Therefore this case happens in the cases (h1) and (h7). In this event AFE is a 30-60-90-degree triangle and the triangle made by m, EG, DA is equilateral (see Figures 36 and 37). The triangle made by CD, CE and the reflection of CD in m is also equilateral.

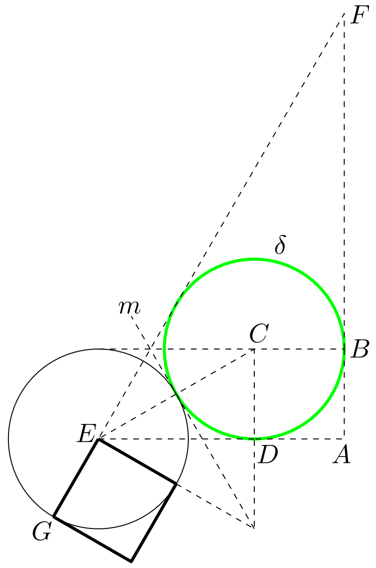

Figure 36: (h1)

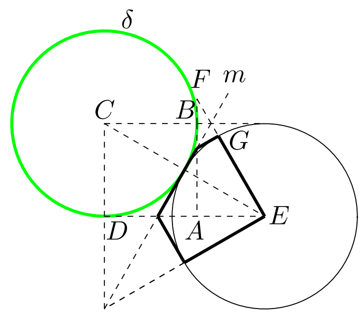

Figure 37: (h7)

8. Conclusion

Haga's fold is generalized, and the recent generalization of Haga's theorems holds for the generalized Haga's fold, which is derived as a very special case from a theorem on incircles and excircles of triangles. There are several simple relationships between the incircles and the excircles of the three similar triangles AFE, GFH and DEI in the figure made by the generalized Haga's fold.

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Tohoku Univ. WDB is short for Tohoku University Wasan Material Database.