Sangaku Journal of Mathematics (SJM) ©SJM
ISSN 2534-9562
Volume 3 (2019), pp.1–2
Received 28 December 2018. Published on-line 11 January 2019
web: http://www.sangaku-journal.eu/

A Pair of Archimedean Incircles

[a]Le Viet An and [b]Emmanuel Antonio José García
[a]No. 15, Alley 2, Ngoc Anh Hamlet, Phu Thuong Ward, Phu Vang District,
Thua Thien Hue, Vietnam.
e-mail: levietan.spt@gmail.com
[b] Universidad Dominicana O&M, Ave. Independencia, 200, Santo Domingo,
Dominican Republic.
e-mail: emmanuelgeogarcia@gmail.com

Abstract. We show that the incircles of a pair of triangles constructed from the arbelos are Archimedean circles.

Keywords. arbelos, Archimedean circles.

Mathematics Subject Classification (2010). 51-04, 68T01, 68T99.

An arbelos is a plane region bounded by three semicircles with three apexes such that each corner of the semicircles is shared with one of the others (connected), all on the same side of a straight line (the baseline) that contains their diameters. One of the properties of the arbelos noticed and proved by Archimedes in his Book of Lemmas is that the two small circles inscribed into two pieces of the arbelos cut off by the line perpendicular to the baseline through the common point of the two small semicircles are congruent. The circles have been known as Archimedes' twin circles. For a point C on the segment AB, let us consider an arbelos formed by the three semicircles (AC), (BC) and (AB), where (AB) denotes the semicircle with diameter AB. Let a and b be the radii of (AC) and (BC), respectively. The radius of Archimedes' twin circles equals $\frac{ab}{a+b}$. Circles of the same radius are said to be Archimedean. In this note we give a pair of Archimedean circles which we hope to be new. A similar construction can be found in [1].

Theorem 1. *Let D, E and F be the centers of semicircles (AC), (BC) and (DE), respectively, C being a point on segment AB. Let G and H be the points where semicircle (DE) meet (AC) and (BC), respectively. Call I the intersection of (AC) and the perpendicular line to FG at G. Similarly, construct J. Call K the intersection of the common tangent lines to (AC) at I and G. Similarly, construct M. The incircles of $\triangle GIK$ and $\triangle HJM$ are Archimedean twins.*

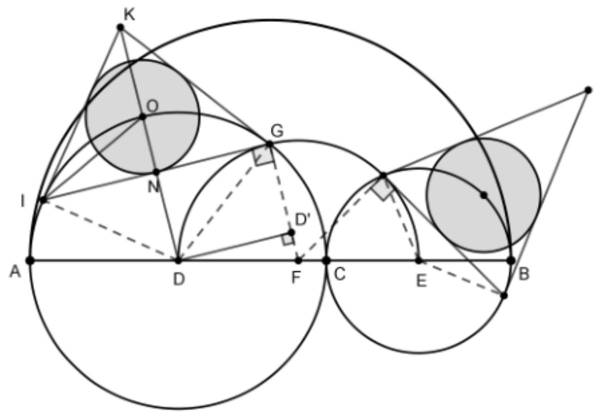

Figure 1. A pair of Archimedean incircles

Proof. Let O be the point of intersection of DK and the angle bisector of $\angle GIK$ (see Figure 1). Notice that O is the incenter of $\triangle GIK$. Let N be the point of intersection of DK and GI. As $DGKI$ is a right kite, DK is a perpendicular bisector of diagonal GI, at N (also the point of tangency of GI and the incircle of $\triangle GIK$). Clearly, $\angle IOD = \angle ION = 90° - \frac{\angle GIK}{2}$. But $\angle DIO = 90° - \frac{\angle GIK}{2}$. Hence, $\triangle DIO$ is isosceles with $DI = DO$. This proves that O lies on the semicircle (AC). Let D' be the orthogonal projection of D onto FG and let x be the inradius of $\triangle GIK$. We have $DN = D'G = a-x$, $NG = DD' = \sqrt{2ax - x^2}$ and $DF = \frac{a+b}{2}$. Focusing on $\triangle DD'F$, by the Pythagorean theorem, we get

$$\left(\frac{a+b}{2} - (a-x)\right)^2 + 2ax - x^2 = \left(\frac{a+b}{2}\right)^2.$$

Solving this equation for x, we get

$$x = \frac{ab}{a+b}.$$

□

Similarly we can get one more Archimedean circle for the semicircle (BC).

References

[1] E. A. J. Garcia, Another Archimedean Circle in an Arbelos, Forum Geom., **15** (2015) 127–128.