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More Relationships Between Six Circles

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Abstract. If P is a point inside $\triangle ABC$, then the cevians through P divide $\triangle ABC$ into smaller triangles of various sizes. We give theorems about the relationships between the radii of various circles associated with these triangles.

Keywords. triangle geometry, incircles, cevians, cevasix configuration.

Mathematics Subject Classification (2010). 51M04.

1. Introduction

Wasan Geometers would often study a configuration consisting of a number of triangles and then inscribe circles in some or all of these triangles. They would then look for a relationship between the radii of these circles.

For example, one such configuration consists of a quadrilateral inscribed in a circle with the diagonals drawn, as shown in Figure 1.

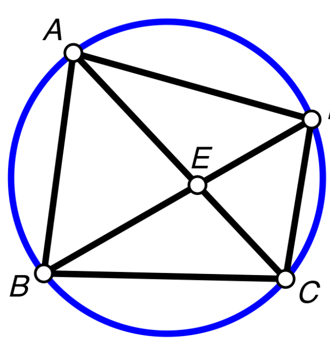

Figure 1.

A well-known sangaku written on a wooden tablet in 1800 in the Yamagata prefecture asked for the relationship between the circles inscribed in triangles ABC, BCD, CDA, and DAB. See [3], [4, p. 43], and [9].

Using the same configuration, a tablet in 1793 in the Fukusima prefecture asked for the relationship between the radius of the original circle and the circles inscribed in the triangles ABE, BCE, CDE, and DEA. See [1], [4, p. 45], and [6].

In this paper, we will do the same thing for a different configuration (shown in Figure 2). Let P be any point inside a triangle ABC. The cevians through P divide $\triangle ABC$ into six smaller triangles. Relationships between the radii of the circles inscribed in triangles APF, BPF, BPD, CPD, CPE, and APE were found in [8]. We will describe other sets of six circles associated with this configuration. Then we will find relationships between the radii of these six circles.

If X and Y are points, then we use the notation XY to denote either the line segment joining X and Y or the length of that line segment, depending on the context. The notation [XYZ] denotes the area of $\triangle XYZ$. The notation $O(r)$ denotes the circle with center O and radius r.

2. Side Triangles

Let P be any point inside $\triangle ABC$ and let AD, BE, CF be the cevians through P. The cevians divide the sides of $\triangle ABC$ into six segments numbered from 1 to 6 as shown in Figure 2.

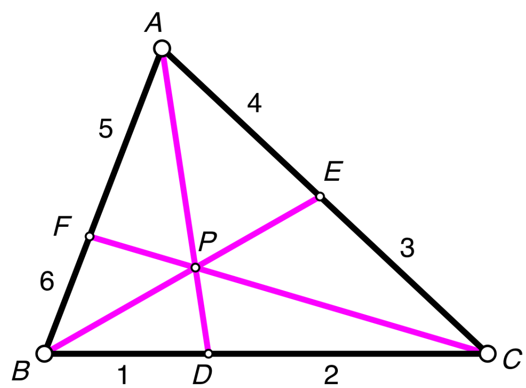

Figure 2. segment numbering

Let triangle T_i be the triangle with one side being the segment numbered i and opposite vertex the vertex of $\triangle ABC$ opposite this side. The six triangles are shown in Figure 3. These triangles will be called "side triangles" of $\triangle ABC$.

Theorem 2.1. *Let H be the orthocenter of $\triangle ABC$ and suppose H lies inside $\triangle ABC$. The altitudes through H form six triangles with the sides of $\triangle ABC$ named T_1 through T_6 as shown in Figure 3. Let r_i be the radius of the circle inscribed in triangle T_i. Then*

$$r_1r_3r_5 = r_2r_4r_6.$$

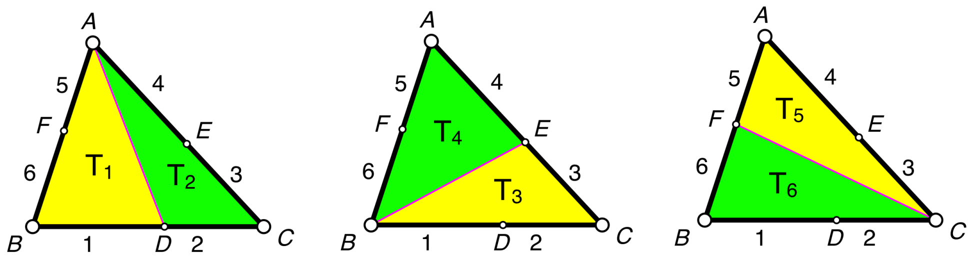

FIGURE 3. triangle numbering

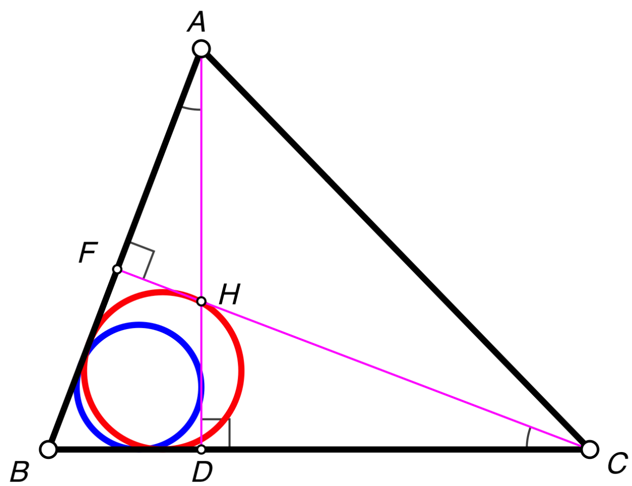

FIGURE 4. $\triangle ABD \sim \triangle CBF$

Proof. Note that $\triangle ABD$ is similar to $\triangle CBF$. Thus, the figure consisting of T_1 and its incircle is similar to the figure consisting of T_6 and its incircle (Figure 4). Corresponding parts of similar figures are in proportion, so $r_1/r_6 = AB/BC$. Similarly, $r_3/r_2 = BC/CA$ and $r_5/r_4 = CA/AB$. Therefore,

$$\frac{r_1 r_3 r_5}{r_2 r_4 r_6} = \frac{r_1}{r_6} \cdot \frac{r_3}{r_2} \cdot \frac{r_5}{r_4} = \frac{AB}{BC} \cdot \frac{BC}{CA} \cdot \frac{CA}{AB} = 1,$$

and the result follows. □

Theorem 2.2. *Let M be the centroid of $\triangle ABC$. The medians through M form six triangles with the sides of $\triangle ABC$ named T_1 through T_6 as shown in Figure 3. Let r_i be the radius of the circle inscribed in triangle T_i. Then*

$$\frac{1}{r_1} + \frac{1}{r_3} + \frac{1}{r_5} = \frac{1}{r_2} + \frac{1}{r_4} + \frac{1}{r_6}.$$

Proof. Let K_i be the area of T_i and let K be the area of $\triangle ABC$. Since a median divides a triangle into two smaller triangles of equal area,

$$K_1 = K_2 = K_3 = K_4 = K_5 = K_6 = K/2.$$

Let s_i be the semiperimeter of T_i. The formula for the incircle of a triangle is K/s where K is the area of the triangle and s is its semiperimeter. Calculating the sum on the left side gives

$$\begin{aligned}\frac{1}{r_1}+\frac{1}{r_3}+\frac{1}{r_5} &= \frac{AB+BD+AD}{K}+\frac{BC+CE+BE}{K}+\frac{CA+AF+CF}{K}\\ &= \frac{(AB+BC+CA)+(BD+CE+AF)+(AD+BE+CF)}{K}\\ &= \frac{\frac{3}{2}(AB+BC+CA)+(AD+BE+CF)}{K}\end{aligned}$$

since $BD = AB/2$, etc. Calculating the sum on the right side,

$$\frac{1}{r_2}+\frac{1}{r_4}+\frac{1}{r_6},$$

gives the same result, so the two sums are equal. □

Suppose the excircles of $\triangle ABC$ touch the sides BC, CA, and AB at points D, E, and F, respectively as shown in Figure 5. Then the cevians AD, BE, and CF, meet at a point, N_a, known as the Nagel point of the triangle [2, p. 160]. We will shortly find the relationship between the radii of the incircles of the T_i when P is the Nagel point of $\triangle ABC$; but first we need a lemma.

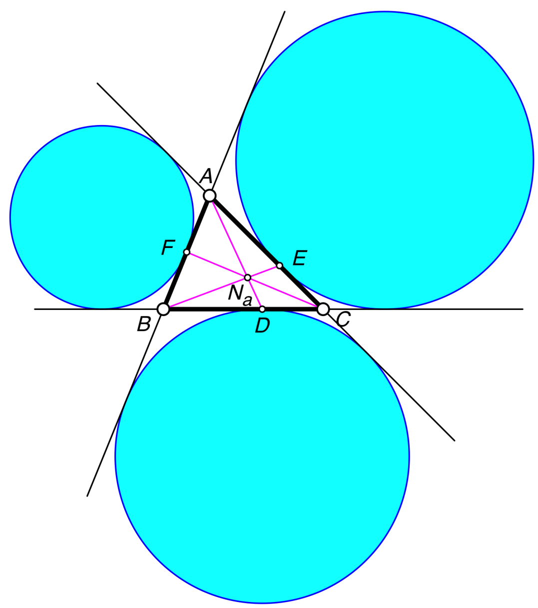

FIGURE 5. Nagel point

Lemma 2.3. *Let D be the contact point of the excircle of $\triangle ABC$ with side BC (Figure 6). The incircle of $\triangle ABD$ has radius r_1. The incircle of $\triangle ADC$ has radius r_2. Then*

$$\frac{r_1}{r_2}=\frac{BD}{DC}.$$

Proof. Using the formula for the radius of an incircle, we have

$$(1) \qquad r_1=\frac{2[ABD]}{AB+BD+AD} \quad \text{and} \quad r_2=\frac{2[ADC]}{AD+DC+CA}.$$

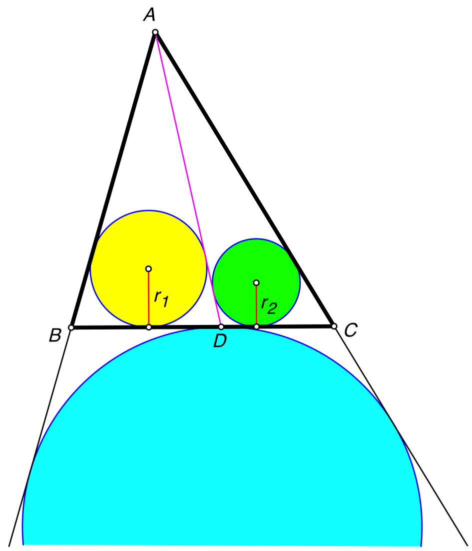

FIGURE 6. $r_1/r_2 = BD/DC$

By a well-known property of the excircle of a triangle [2, p. 88], $BD = s - AB$ and $DC = s - CA$, where $s = (AB+BC+CA)/2$. Thus $CA+DC = AB+BD$, which implies $AB + BD + AD = AD + DC + CA$. Therefore, the two denominators in equation (1) are equal. Hence $r_1/r_2 = [ABD]/[ADC]$. Since triangles ABD and ADC have the same altitude from A, the ratio of their areas will be proportional to the ratio of their bases. Consequently,

$$\frac{r_1}{r_2} = \frac{[ABD]}{[ADC]} = \frac{BD}{DC}$$

as claimed. □

We can now prove the following theorem.

Theorem 2.4. *Let N_a be the Nagel point of $\triangle ABC$. The cevians through N_a form six triangles with the sides of $\triangle ABC$ named T_1 through T_6 as shown in Figure 3. Let r_i be the radius of the circle inscribed in triangle T_i. Two of these circles are shown in Figure 7. Then $r_1r_3r_5 = r_2r_4r_6$.*

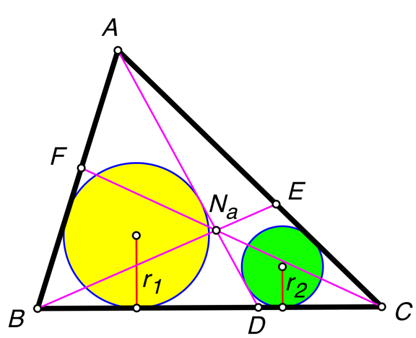

FIGURE 7. $r_1r_3r_5 = r_2r_4r_6$

Proof. By Lemma 2.3,

$$\frac{r_1}{r_2}\cdot\frac{r_3}{r_4}\cdot\frac{r_5}{r_6}=\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}.$$

The expression on the right is equal to 1 by Ceva's Theorem. Thus, we have $r_1r_3r_5 = r_2r_4r_6$. □

We have found simple relationships between the incircles of the six side triangles formed by the cevians through a point P inside the triangle when P is the orthocenter, centroid, and Nagel point of $\triangle ABC$. No such simple relationships were found when P is the circumcenter, Feuerbach point, Gergonne point, incenter, nine-point center, or Spieker center of the triangle. By this we mean that none of the following relationships hold for all such triangles.

$$\begin{gathered}
r_1+r_3+r_5=r_2+r_4+r_6,\\
r_1r_3r_5=r_2r_4r_6,\\
r_1^2+r_3^2+r_5^2=r_2^2+r_4^2+r_6^2,\\
r_1r_3+r_3r_5+r_5r_1=r_2r_4+r_4r_6+r_6r_2,\\
\frac{1}{r_1}+\frac{1}{r_3}+\frac{1}{r_5}=\frac{1}{r_2}+\frac{1}{r_4}+\frac{1}{r_6},\\
\frac{1}{r_1^2}+\frac{1}{r_3^2}+\frac{1}{r_5^2}=\frac{1}{r_2^2}+\frac{1}{r_4^2}+\frac{1}{r_6^2}.
\end{gathered}$$

We now consider the circumcircles of the side triangles.

Theorem 2.5. *Let P be any point inside $\triangle ABC$. The cevians through P form six triangles with the sides of $\triangle ABC$ named T_1 through T_6 as shown in Figure 3. Let R_i be the radius of the circle circumscribed about triangle T_i. Then*

$$R_1R_3R_5 = R_2R_4R_6.$$

Proof. We use of The Extended Law of Sines which states that if a, b, and c are the lengths of the sides of a triangle opposite angles A, B, and C, then

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=2R$$

where R is the circumradius of $\triangle ABC$. Thus,

$$2R_1=\frac{AD}{\sin\angle ABC},$$

with similar expressions for the other R_i. Hence

$$R_1R_3R_5=\frac{1}{8}\frac{AD}{\sin\angle ABC}\cdot\frac{BE}{\sin\angle BCA}\cdot\frac{CF}{\sin\angle CAB}$$

and

$$R_2R_4R_6=\frac{1}{8}\frac{AD}{\sin\angle BCA}\cdot\frac{BE}{\sin\angle CAB}\cdot\frac{CF}{\sin\angle ABC}.$$

The two results are equal, so $R_1R_3R_5 = R_2R_4R_6$. □

Theorem 2.6. *Let H be the orthocenter $\triangle ABC$ and assume H lies inside $\triangle ABC$. The altitudes through H form six triangles with the sides of $\triangle ABC$ named T_1 through T_6 as shown in Figure 3. Let R_i be the radius of the circle circumscribed about triangle T_i. Then*

$$R_1 = R_4, \qquad R_2 = R_5, \qquad R_3 = R_6.$$

Proof. It suffices to show that $R_1 = R_4$. Triangles T_1 and T_4 are right triangles with the same hypotenuse, AB. But the circumradius of a right triangle is equal to half the length of the hypotenuse, so $R_1 = R_4$. □

3. Small Wedge Triangles

Let AD, BE, and CF be the three cevians through a point P inside $\triangle ABC$. Extend the cevians to the circumcircle of $\triangle ABC$ to get line segments AD', BE', and CF', where D', E', and F' lie on the circumcircle. Then three new triangles are formed with segment BD' as one side and the other two sides lying along sides or cevians of $\triangle ABC$. The three new triangles are shown in Figure 8.

We call these triangles wedge triangles. From left to right, the figure shows a small wedge triangle, a medium wedge triangle, and a large wedge triangle.

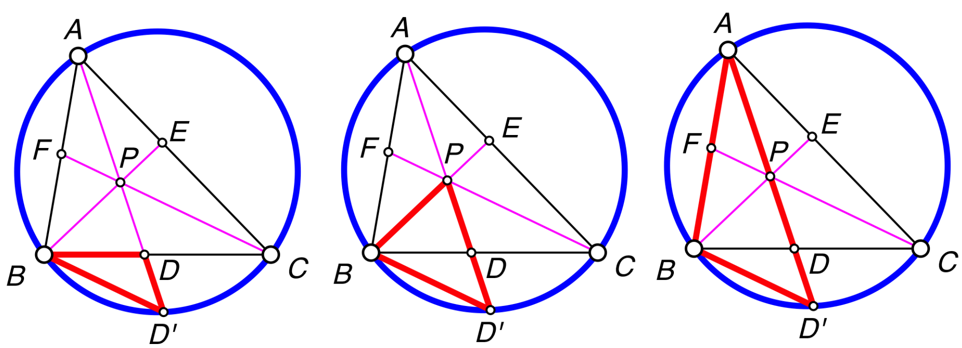

FIGURE 8. small, medium, and large wedge triangles

The points A, E', C, D', B, and F' form a hexagon inscribed in the circumcircle. There are six wedge triangles of each type associated with the sides of this hexagon. We will investigate the relationship between the radii of the circles inscribed in (or circumscribed about) each set of six wedge triangles.

We start with the six small wedge triangles.

Theorem 3.1. *Let H be the orthocenter of $\triangle ABC$ and assume H lies inside $\triangle ABC$. The altitudes through H, AD, BE, and CF are extended to meet the circumcircle of $\triangle ABC$ at points D', E', and F', respectively. Let $O_i(r_i)$ be the incircles of the six small wedge triangles formed as shown in Figure 9. Then*

$$r_1 r_3 r_5 = r_2 r_4 r_6.$$

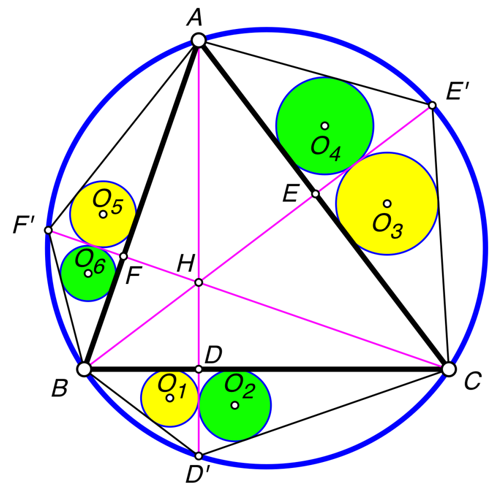

FIGURE 9. $r_1r_3r_5 = r_2r_4r_6$

Proof. Note that $m\angle BD'A = m\angle BE'A$ since both angles subtend the same arc. Angles BDD' and AEE' are right angles. Therefore, $\triangle BDD' \sim \triangle AEE'$. Thus, the figure consisting of $\triangle BDD'$ and its incircle is similar to the figure consisting of $\triangle AEE'$ and its incircle. Corresponding parts of similar figures are in proportion, so $r_1/r_4 = BD/EA$. Similarly $r_3/r_6 = CE/FB$ and $r_5/r_2 = AF/DC$. Therefore,

$$\frac{r_1r_3r_5}{r_2r_4r_6} = \frac{r_1}{r_4}\cdot\frac{r_3}{r_6}\cdot\frac{r_5}{r_2} = \frac{BD}{EA}\cdot\frac{CE}{FB}\cdot\frac{AF}{DC} = \frac{BD\cdot CE\cdot AF}{DC\cdot EA\cdot FB}.$$

This last fraction is equal to 1 by Ceva's Theorem, so $r_1r_3r_5 = r_2r_4r_6$. □

Theorem 3.2. *Let N_a be the Nagel point of $\triangle ABC$. The cevians through N_a, AD, BE, and CF are extended to meet the circumcircle of $\triangle ABC$ at points D', E', and F', respectively. Let $O_i(r_i)$ be the incircles of the six small wedge triangles formed as shown in Figure 10. Then $r_1 = r_2$, $r_3 = r_4$, and $r_5 = r_6$.*

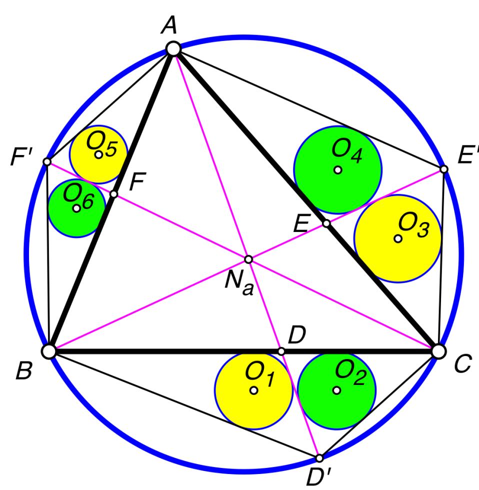

FIGURE 10. $r_1 = r_2,\ r_3 = r_4,\ r_5 = r_6$

Proof. It suffices to show that $r_1 = r_2$. Let K_1 be the area of $\triangle BDD'$ and let s_1 be its semiperimeter. Define K_2 and s_2 analogously. From the formula for the inradius of a triangle, we have

$$\frac{r_1}{r_2} = \frac{K_1/s_1}{K_2/s_2} = \frac{K_1/(BD + DD' + BD')}{K_2/(DC + DD' + CD')}.$$

The ratio of the areas of two triangles with the same altitude is equal to the ratio of their bases. So

$$\frac{r_1}{r_2} = \frac{BD/(BD + DD' + BD')}{DC/(DC + DD' + CD')} = \frac{BD(DC + DD' + CD')}{DC(BD + DD' + BD')}.$$

Dividing numerator and denominator by DD' gives

$$\frac{r_1}{r_2} = \frac{BD(\frac{DC}{DD'} + 1 + \frac{CD'}{DD'})}{DC(\frac{BD}{DD'} + 1 + \frac{BD'}{DD'})}.$$

Angles $\angle BCD'$ and $\angle BAD$ are congruent because they intercept the same arc. The same is true for $\angle CD'D$ and $\angle ABD$. Thus, $\triangle CDD' \sim \triangle ADB$ and hence $CD'/DD' = AB/BD$ and $DC/DD' = AD/BD$. Similarly, $BD'/DD' = CA/DC$ and $BD/DD' = AD/DC$. Therefore,

$$\frac{r_1}{r_2} = \frac{BD(\frac{AD}{BD} + 1 + \frac{AB}{BD})}{DC(\frac{AD}{DC} + 1 + \frac{CA}{DC})} = \frac{AD + BD + AB}{AD + DC + CA}.$$

But, as noted in the proof of Lemma 2.3, $AD + BD + AB = AD + DC + CA$. Hence, the fraction on the right is equal to 1, so $r_1 = r_2$. □

One aspect of Sangaku problems is that they often conclude with some simple relationship between one or more incircles that is not immediately apparent. I find the result especially pretty when it concludes that two incircles are congruent. The following example is an adaptation of problem 2 from [7], which is based on an 1881 sangaku hung in the Yamagata prefecture, [5].

Theorem 3.3. *In Figure 11, $ABCD$ is a square, CFG is an equilateral triangle, and E is the midpoint of CF. Then the two yellow incircles are congruent.*

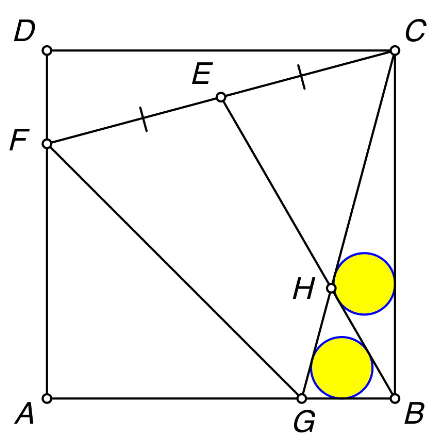

FIGURE 11. yellow circles are congruent

Wasan geometers often referred to medians and altitudes of triangles. However, the concept of the Nagel point of a triangle was probably unknown to the Japanese mathematicians during the Edo period. So Theorem 3.2 is unlikely to be something they would have discovered. However, Theorem 3.2 can be reformulated as follows to put it in a form that would have been appropriate to have been inscribed on a wooden tablet and hung in a temple as a pretty result.

Theorem 3.4 (reformulation of Theorem 3.2)**.** *In Figure 12, D is a point on side BC of $\triangle ABC$ such that $AB + BD = AC + CD$. Then the two yellow incircles are congruent.*

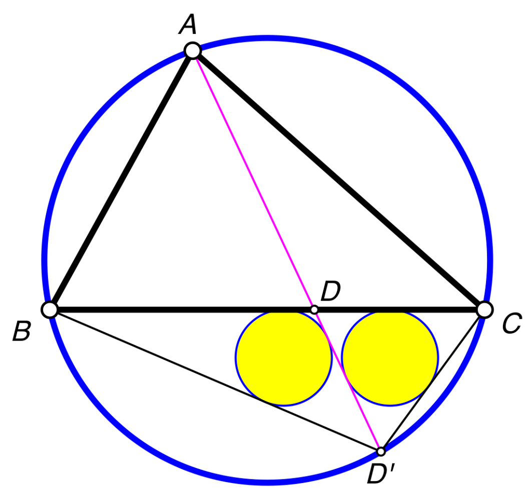

FIGURE 12. yellow circles are congruent

We have found simple relationships between the incircles of the six small wedge triangles formed by the cevians through a point P inside the triangle extended to the circumcircle when P is the orthocenter or the Nagel point of $\triangle ABC$. No such simple relationships were found when P is the centroid, circumcenter, Feuerbach point, Gergonne point, incenter, nine-point center, or Spieker center of the triangle.

We now consider the circumcircles of the small wedge triangles.

Theorem 3.5. *Let P be any point inside $\triangle ABC$. The cevians through P, AD, BE, and CF are extended to meet the circumcircle of $\triangle ABC$ at points D', E', and F', respectively. Let $O_i(R_i)$ be the circumcircles of the six small wedge triangles formed as shown in Figure 13. Then $R_1R_3R_5 = R_2R_4R_6$.*

Proof. By the Extended Law of Sines, we have

$$2R_1 = \frac{BD}{\sin BD'A} \quad \text{and} \quad 2R_4 = \frac{AE}{\sin \angle BE'A}.$$

But $m\angle BD'A = m\angle BE'A$ because both angles subtend the same arc of the circumcircle of $\triangle ABC$. Therefore, $R_1/R_4 = BD/EA$. Similarly, $R_2/R_5 = DC/AF$

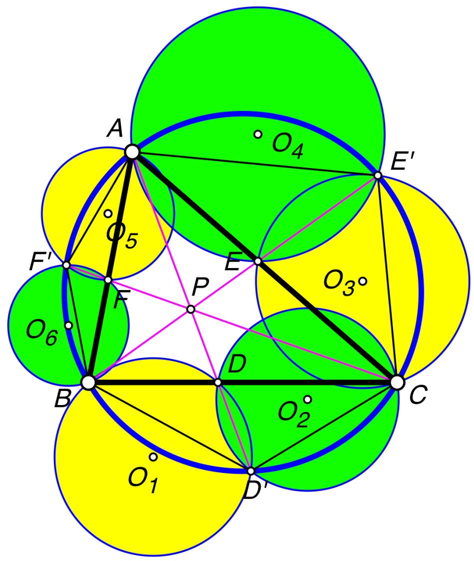

FIGURE 13. $R_1R_3R_5 = R_2R_4R_6$

and $R_3/R_6 = CE/FB$. Therefore

$$\frac{R_1R_3R_5}{R_2R_4R_6} = \frac{R_1}{R_4} \cdot \frac{R_3}{R_6} \cdot \frac{R_5}{R_2} = \frac{BD}{EA} \cdot \frac{CE}{FB} \cdot \frac{AF}{DC} = \frac{BD \cdot CE \cdot AF}{DC \cdot EA \cdot FB}.$$

This last fraction is equal to 1 by Ceva's Theorem, so $R_1R_3R_5 = R_2R_4R_6$. □

4. MEDIUM WEDGE TRIANGLES

Theorem 4.1. *Let P be any point inside $\triangle ABC$. The cevians through P, AB, BD, and CF are extended to meet the circumcircle of $\triangle ABC$ at points D', E', and F', respectively. Circles $O_i(r_i)$ are inscribed in the six medium wedge triangles formed as shown in Figure 14. Then $r_1r_3r_5 = r_2r_4r_6$.*

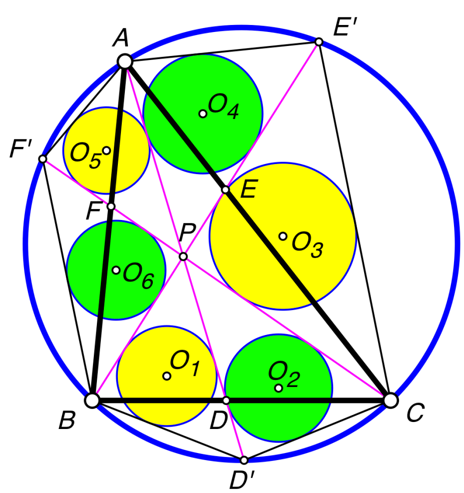

FIGURE 14. $r_1r_3r_5 = r_2r_4r_6$

Proof. Note that $\angle AE'B$ and $\angle AD'B$ both subtend the same arc and are therefore congruent. Also, $\angle BPD' = \angle APE'$ because they are vertical angles. Therefore $\triangle BPD' \sim \triangle APE'$. Thus, the figure consisting of $\triangle BPD'$ and its incircle is similar to the figure consisting of $\triangle APE'$ and its incircle. Corresponding parts of similar figures are in proportion, so

$$\frac{r_1}{r_4} = \frac{BP}{AP}.$$

In the same manner,

$$\frac{r_5}{r_2} = \frac{AP}{CP} \qquad \text{and} \qquad \frac{r_3}{r_6} = \frac{CP}{BP}.$$

Consequently,

$$\frac{r_1 r_3 r_5}{r_2 r_4 r_6} = \frac{r_1}{r_4} \cdot \frac{r_3}{r_6} \cdot \frac{r_5}{r_2} = \frac{BP}{AP} \cdot \frac{CP}{BP} \cdot \frac{AP}{CP} = 1$$

and the result follows. □

5. Large Wedge Triangles

Theorem 5.1. *Let O be the circumcenter of $\triangle ABC$. The cevians through O, AD, BE, and CF are extended to meet the circumcircle of $\triangle ABC$ at points D', E', and F', respectively. Let $O_i(r_i)$ be the incircles of the six large wedge triangles formed (two of these circles are shown in Figure 15). Then*

$$R_1 = R_4, \quad R_2 = R_5, \quad R_3 = R_6.$$

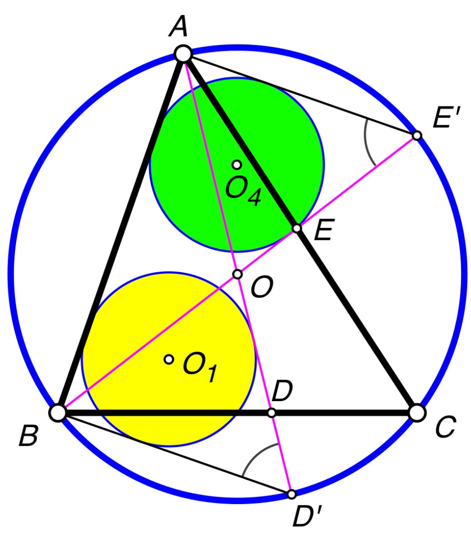

Figure 15. $R_1 = R_4$

Proof. It suffices to show that $R_1 = R_4$. Note that AD' and BE' are diameters of the circumcircle, so $\angle BAE'$ and $\angle ABD'$ are right angles. Also, $m\angle BD'A = m\angle BE'A$ since both angles subtend the same arc. Segment AB is common to both triangles. Hence $\triangle ABD' \cong \triangle BAE'$. Thus, their incircles are also congruent. □

We have found a simple relationship between the incircles of the six large wedge triangles formed by the cevians through a point P inside the triangle extended to the circumcircle when P is the circumcenter of $\triangle ABC$. No such simple relationships were found when P is the centroid, Feuerbach point, Gergonne point, incenter, Nagel point, nine-point center, orthocenter, or Spieker center of the triangle.

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