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A note on the arbelos in Wasan geometry, Matsuda's problem

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Abstract. We generalize the sangaku problem involving an arbelos proposed by Matsuda and show the existence of several non-Archimedean congruent circles.

Keywords. sangaku, arbelos, non-Archimedean congruent circles

Mathematics Subject Classification (2010). 01A27, 51M04, 51N20.

1. Introduction

We consider an arbelos appeared in Wasan geometry and consider an arbelos formed by three semicircles α, β and γ of diameters AO, BO and AB, respectively for a point O on the segment AB (see Figure 1). The radical axis of α and β is called the axis. Let I be the point of intersection of the axis and γ. For a circle δ passing through I and its reflection in the line AO and intersecting α, the incircle of the curvilinear triangle made by α, γ and δ is denoted by ε_1, and the incircle of the curvilinear triangle made by α, δ and IO is denoted by ε_2. Let C be the center of α. In this note we consider the following problem in a sangaku in Sendai proposed by Matsuda (松田運蔵) [2] (see Figure 2).

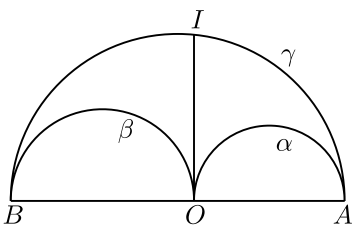

Figure 1.

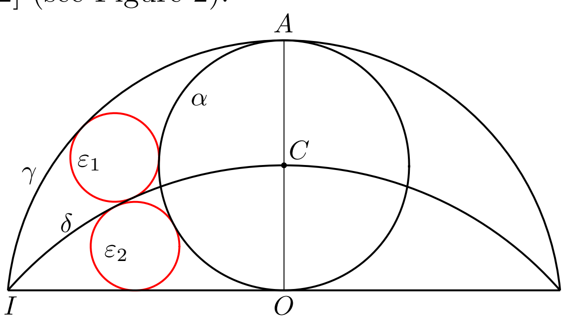

Figure 2.

Problem 1. If ε_1 and ε_2 are congruent, show that δ passes through the point C.

2. Generalization

Let a and b be the radii of α and β, respectively. We use a rectangular coordinate system with origin O such that the farthest point on α from AB has coordinates (a, a). Problem 1 is generalized as follows (see Figure 3).

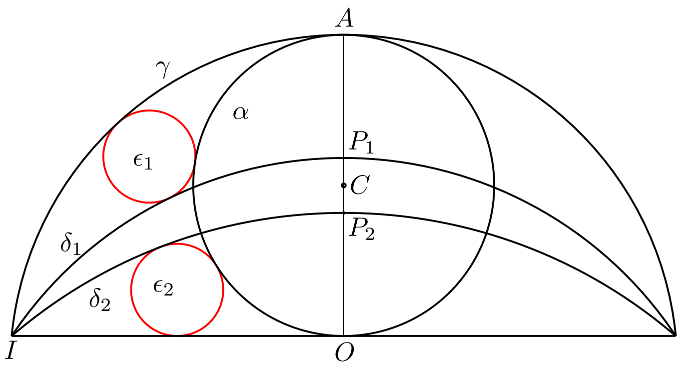

Figure 3.

Theorem 1. *For a point P_i $(i = 1, 2)$ on the segment AO, let δ_i be the circle passing through the points P_i, I and its reflection in the line AO. Then the incircle of the curvilinear triangle made by α, γ and δ_1 and the incircle of the curvilinear triangle made by α, δ_2 and the line IO are congruent if and only if C is the midpoint of P_1P_2.*

Proof. We assume that ϵ_i is one of the incircles touching δ_i and has radius r_i. The circle δ_i has an equation $(x - 2p_i)(x + 2q_i) + y^2 = 0$ for $p_i > 0$ and $q_i > 0$. Since δ_i pass through the point I, whose y-coordinate is $2\sqrt{ab}$, we get $p_iq_i = ab$. Let (x_i, y_i) be the coordinates of the center of ϵ_i, where $x_2 = r_2$. Since the distances from the center of ϵ_1 to the centers of α, γ and δ_1 are $a + r_1$, $a + b - r_1$ and $p_1 + q_1 + r_1$, respectively, we have $(a-x_1)^2+y_1^2 = (a+r_1)^2$, $(a-b-x_1)^2+y_1^2 = (a+b-r_1)^2$ and $(p_1 - q_1 - x_1)^2 + y_1^2 = (p_1 + q_1 + r_1)^2$. Similarly we have $(a - r_2)^2 + y_2^2 = (a + r_2)^2$ and $(p_2 - q_2 - r_2)^2 + y_2^2 = (p_2 + q_2 - r_2)^2$. Eliminating q_i by $p_iq_i = ab$ and x_1, y_1 and y_2 from the five equations and solving the resulting equations for r_i, we have

$$r_1 = \frac{b(a - p_1)}{a + b - p_1}, \quad \text{and} \quad r_2 = \frac{bp_2}{b + p_2}. \tag{1}$$

Then

$$r_1 - r_2 = \frac{b^2(a - p_1 - p_2)}{(a + b - p_1)(b + p_2)}$$

implies that $r_1 = r_2$ is equivalent to $a = p_1 + p_2$. Since P_i has coordinates $(2p_i, 0)$, the theorem is proved. □

We now return to the figure of Problem 1.

Corollary 1. *The circles ε_1 and ε_2 are congruent if and only if the circle δ passes through the point C. In this event the following statements hold.*
(i) *If D is the remaining point of intersection of δ and AB, then $|DO| = 4b$.*
(ii) *The common radius of ε_i equals $ab/(a + 2b)$.*
(iii) *One of the internal common tangents of ε_1 and ε_2 is perpendicular to AB.*
(iv) *The circle of center A passing through O touches ε_1 internally.*
(v) *Let l_1 be the common tangent of ε_1 and ε_2 in* (iii). *If (AD) is the semicircle*

of diameter AD lying on the same side of AB as γ, and meets the axis in a point J, then γ and the line AJ meet on l_1.
(vi) *Let l_2 be the remaining tangent of ε_1 perpendicular to AB. Then the circle stated in* (iv) *and* (AD) *meet on l_2, also α and AJ meet on l_2.*
(vii) *The circle touching the tangent of β from C and the axis at O from the side opposite to B is congruent to ε_i.*

Proof. We assume that δ passes through C. Then δ has an equation $(x-2p)(x+2q)+y^2=0$ for $p>0$ and $q>0$, where $p=a/2$. While we have $pq=ab$, for δ passes through I. Hence we get $q=2b$. This proves (i). Let r be the radius of ε_i. The part (ii) follows from $r=bp/(b+p)$ by (1) with $p=a/2$. Let (x_1,y_1) be the coordinates of the center of ε_1. Solving the equation $(a+r)^2-(a-x_1)^2=(a+b-r)^2-(a-b-x_1)^2$ with $r=ab/(a+2b)$, we get $x_1=3r$. This proves (iii). Since $y_1^2=(a+r)^2-(a-x_1)^2=8r(a-r)$, the distance between the center of ε_1 and A equals $2a-r$. This proves (iv). The parts (v) and (vi) are proved by the fact that the point J has y-coordinate $2\sqrt{2ab}$. The part (vii) is proved by the two similar right triangles made by AB, the tangent of β from C and each of the perpendiculars to AB at B and O. □

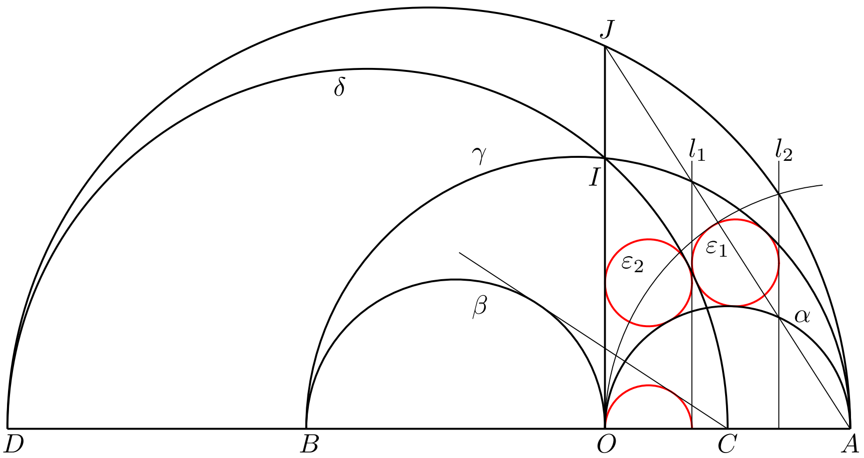

Figure 4.

3. SIX CIRCLES CONGRUENT TO ε_i

We use the next theorem [1, Theorem 1] (see Figures 5 and 6 for Theorem 2(i) and see Figure 7 for Theorem 2(ii)).

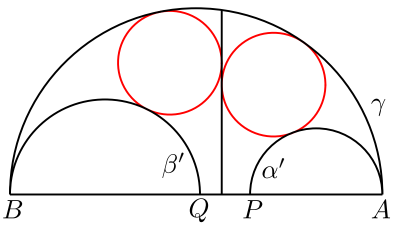

Figure 5.

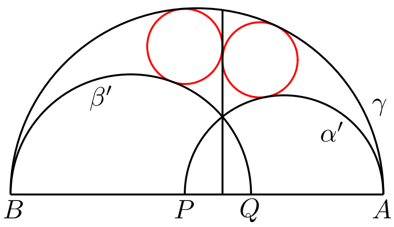

Figure 6.

Theorem 2. *For points P and Q on the line AB, let α' and β' be the semicircles of diameters AP and BQ constructed on the same side of AB as γ, respectively. Let $s=|AQ|/2$ and $t=|BP|/2$. Then the following statements hold.*

(i) *If P and Q lie between A and B, the radius of the circle touching α' (resp. β') externally γ internally and the radical axis of α' and β' from the side opposite to B (resp. A) equals $st/(s+t)$.*
(ii) *If the four points P, B, A and Q lie in this order, then the radius of the circle touching α' (resp. β') internally γ externally and the radical axis of α' and β' from the side opposite to A (resp. B) equals $st/(s+t)$.*

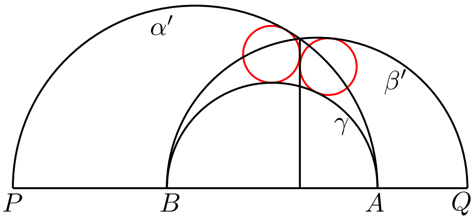

Figure 7.

We still assume that δ passes through C. Since circles of radius $ab/(a+b)$ are said to be Archimedean, ε_1 and ε_2 are non-Archimedean congruent circles of radius $ab/(a+2b)$. If the semicircles (BC) and (CO) are defined similarly as to (AD), the following six circles are congruent to ε_i by Theorem 2, which are denoted by the green circles in Figure 8:

(1) the circle touching γ externally (AD) internally and the axis from the side opposite to B denoted by ι_1,
(2) the circle touching δ externally (AD) internally and the axis from the side opposite to A denoted by ι_2,
(3) the circle touching (BC) externally γ internally and the axis from the side opposite to B denoted by ι_3,
(4) the circle touching (BC) externally δ internally and the axis from the side opposite to A denoted by ι_4,
(5) the circle touching (CO) externally (BC) internally and the axis from the side opposite to B denoted by ι_5,
(6) the circle touching β externally (BC) internally and the axis from the side opposite to A denoted by ι_6.

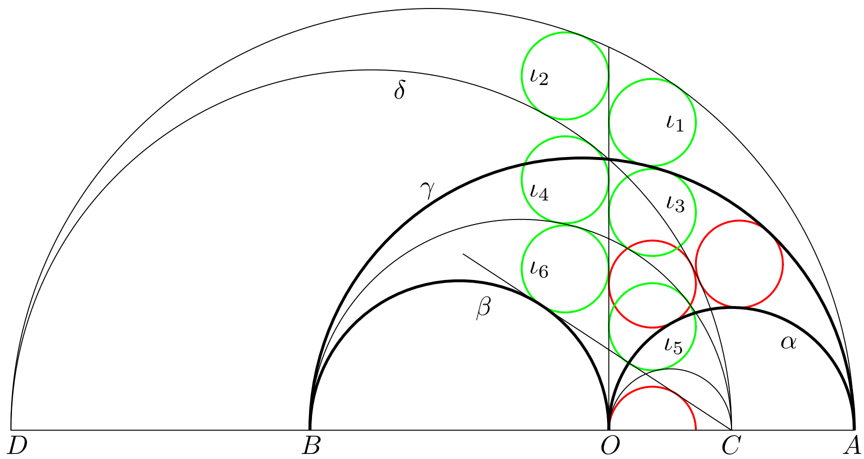

Figure 8.

4. Open problem

For a point P_i $(i = 0, 1, 2, \cdots, n)$ on the segment AO, let δ_i be the circle passing through the points P_i, I and its reflection in the line AO, where $P_0 = O$ and $P_n = A$. Let ε_i be the incircle of the curvilinear triangle made by δ_{i-1}, δ_i and α for $i = 1, 2, \cdots, n$ (see Figure 9). The case $n = 1$ is trivial, where the radius of ε_1 equals the radius of Archimedean circles $ab/(a+b)$. The case $n = 2$ is related to Matsuda's problem, and details are considered in Corollary 1. The following problems for $n \geq 3$ are open.

(i) Determine a condition that the incircles ε_1, ε_2, $\cdots$, ε_n are all congruent.

(ii) In this event find the common radius of ε_i in the terms of a, b and n.

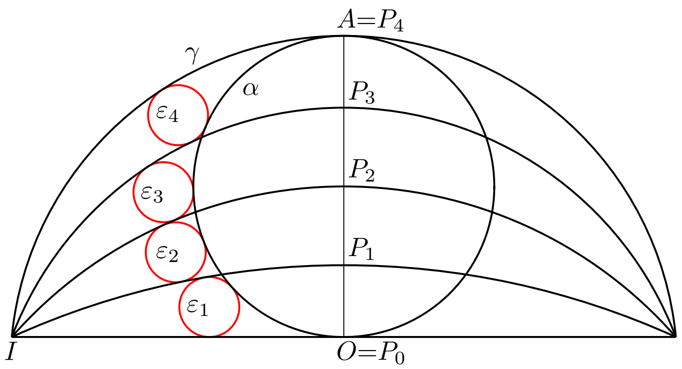

Figure 9: $n = 4$

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