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Diagram 29 in the Appendix of Yojutsu Shindai

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Abstract. We investigate Diagram 29 in the Appendix of Yojutsu Shindai, which contains one hundred Sangaku-like diagrams about tangent circles and lines in a square. While the circles in most of these diagrams fall into three kinds, small, medium, and large, of radii in the ratio 1 : 2 : 4, Diagram 29 is one of the exceptions. We determine the proportions of these radii precisely.

Keywords. square, tangent circle.

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1. Introduction

The Appendix of the remarkable book [2] (see also [1, pp. 189–206]) contains one hundred Sangaku-like diagrams, each consisting of tangent circles and lines in a square. For most of these diagrams, the complete circles in each diagram fall into three kinds, small, medium, and large, of radii in the ratio 1 : 2 : 4. There are exceptions, and Diagram 29 is one of them (see Figure 1). Here, while the small and medium circles have radii in the ratio 1 : 2, the large circles do not have radius 4. The purpose of this note is to determine this radius, and to decide if there are different configurations beginning with a large circle in a corner of the square.

To simplify the algebra, we consider instead the reflection of the diagram about a line joining the midpoints of two opposite sides of the square (see Figure 2), which we label as $OABC$ and has each side of length a. We shall work with a Cartesian coordinate system with origin at O such that B has coordinates (a, a), i.e., $A = (a, 0)$ and $C = (0, a)$. For a given $R > 0$, beginning with a circle $\mathcal{C}_1$ with

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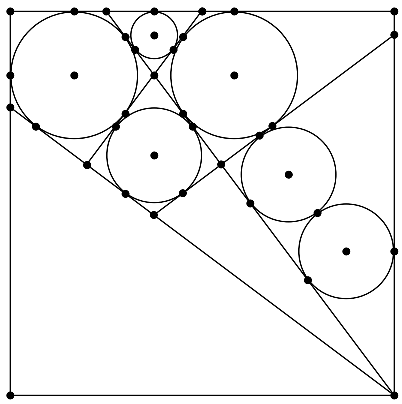

Figure 1: JTG 29

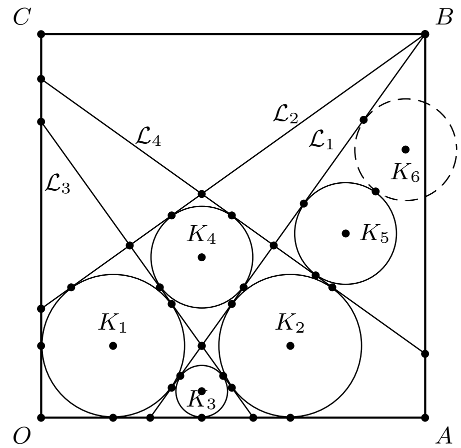

Figure 2: JTG 29$'$

center $K_1 = (R, R)$ and radius $r_1 = R$, we construct, inside the square,
(i) the two tangents $\mathcal{L}_1$, $\mathcal{L}_2$ from B,
(ii) the circle $\mathcal{C}_2$ congruent to $\mathcal{C}_1$ and tangent to both OA and $\mathcal{L}_1$,
(iii) apart from $\mathcal{L}_1$, the second internal common tangent $\mathcal{L}_3$ of the circles $\mathcal{C}_1$ and $\mathcal{C}_2$,
(iv) the circle $\mathcal{C}_3$ tangent to the lines OA, $\mathcal{L}_1$, and $\mathcal{L}_3$,
(v) the incircle $\mathcal{C}_4$ of the triangle bounded by the lines $\mathcal{L}_1$, $\mathcal{L}_2$, and $\mathcal{L}_3$,
(vi) the second external common tangent $\mathcal{L}_4$ of the circles $\mathcal{C}_2$ and $\mathcal{C}_4$.

In this note we consider the possibility of two circles $\mathcal{C}_5$ and $\mathcal{C}_6$ on the same side of $\mathcal{L}_1$, congruent to $\mathcal{C}_4$, tangent to each other and to $\mathcal{L}_1$, so that $\mathcal{C}_5$ is also tangent to $\mathcal{L}_4$, and $\mathcal{C}_6$ to AB.

Notation: For $i = 1, 2, \ldots, 6$, we label the center of the circle $\mathcal{C}_i$ by K_i, and its radius by r_i. Also, for $j = 1, 2, 3, 4$, the point of tangency of $\mathcal{C}_i$ and $\mathcal{L}_j$ is denoted by $T_{i,j}$ (provided that the two objects are tangent to each other). The point of intersection of the lines $\mathcal{L}_i$ and $\mathcal{L}_j$ is denoted by $P_{i,j}$. For the sides of the square, we use x, y, x', y' for the OA, OC, CB, AB respectively. In the diagrams, the points are usually not labelled, but can be easily identified from the tangency or intersecting lines. The point of tangency of a circle and a line is clearly the orthogonal projection of the center of the circle on the line. These are computed using the Lemma below.

Lemma. *The orthogonal projection of $P = (u, v)$ on $\mathcal{L}: \ fx + gy + h = 0$ is the point*

$$\left(\frac{g(gu - fv) - fh}{f^2 + g^2}, \ \frac{-f(gu - fv) - gh}{f^2 + g^2}\right).$$

Proof. This is the point of intersection of the two lines

$$\begin{aligned} fx + gy + h &= 0, \\ -gx + fy + gu - fv &= 0. \end{aligned}$$

□

2. THE CIRCLES $\mathcal{C}_1$ AND $\mathcal{C}_2$

Proposition 1. *Suppose the tangents $\mathcal{L}_1$ and $\mathcal{L}_2$ intersect OA and OC at $P_{1,x}$ and $P_{2,y}$ respectively, with $P_{1,x}A = P_{2,y}C = b < a$, and $BP_{1,x} = c$* (see Figure 3). *Then*

$$R = \frac{a(a-b)}{c+a-b}.$$

Proof. Consider the right triangle $ABP_{1,x}$ with sides $BA = a$, $AP_{1,x} = b$, $P_{1,x}B = c$. Its excircle on the side $AP_{1,x}$ has radius $\frac{ab}{c+a-b}$. The right triangle $OP_{1,y}P_{1,x}$ is similar to $ABP_{1,x}$ with $\frac{OP_{1,x}}{AP_{1,x}} = \frac{a-b}{b}$. Its excircle on the side $OP_{1,x}$ is the circle $\mathcal{C}_1$ (see Figure 3). Therefore,

$$R = \frac{a-b}{b} \cdot \frac{ab}{c+a-b} = \frac{a(a-b)}{c+a-b}.$$

□

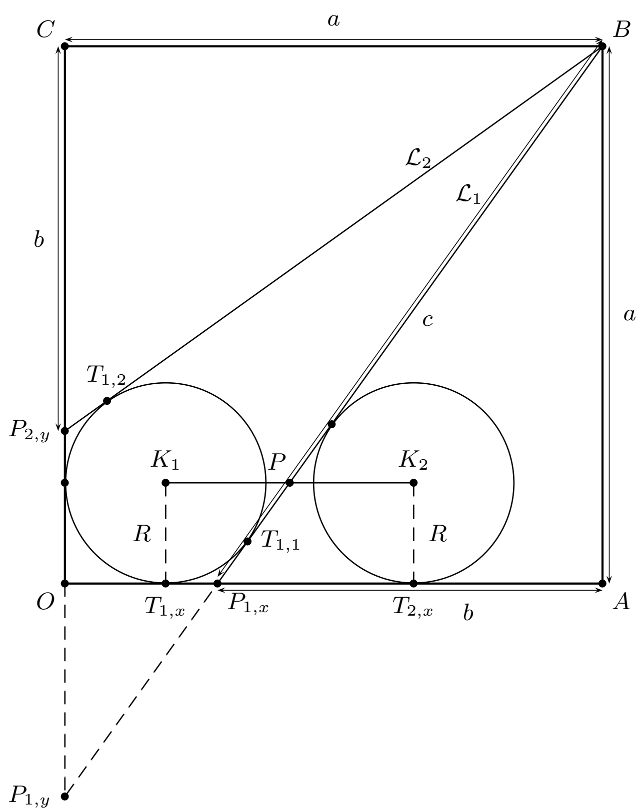

FIGURE 3.

By the coordinates $P_{1,x} = (a-b, 0)$, $P_{2,y} = (0, a-b)$, and $B = (a, a)$, we obtain the equations of the lines

$\mathcal{L}_1$: $$ax - by - a(a-b) = 0$$

and

$\mathcal{L}_2$: $$bx - ay + a(a-b) = 0.$$

From these we find the points of tangency $T_{1,1}$ and $T_{1,2}$. Note that these are symmetric with respect to the line $y = x$.

Let P be the point of intersection of the lines $y = R$ and $\mathcal{L}_1$ (see Figure 3). This is the point

$$P = \left(\frac{a(a-b)+bR}{a}, R\right).$$

Since the point K_2 is the reflection of K_1 in the point P,

$$K_2 = 2P - K_1 = \left(\frac{2a(a-b)-(a-2b)R}{a}, R\right).$$

3. THE CIRCLES $\mathcal{C}_3$ AND $\mathcal{C}_4$

Since $P_{3,x}$ is the reflection of $P_{1,x}$ in the line $x = \frac{a(a-b)+bR}{a}$,

$$P_{3,x} = 2\left(\frac{a(a-b)+bR}{a}, 0\right) - (a-b, 0) = \left(\frac{a(a-b)+2bR}{a}, 0\right).$$

From this, we obtain the points of tangency $T_{2,x}$, $T_{2,1}$, $T_{2,3}$, and $T_{1,3}$ (see Figure 4).

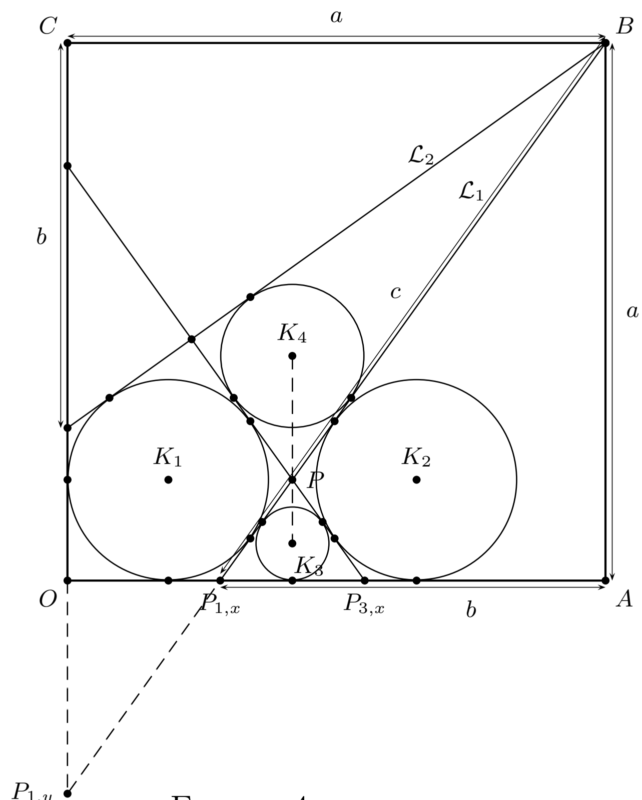

FIGURE 4.

Using the points P and $P_{3,x}$ we obtain the equation of the line $\mathcal{L}_3$:

$$\mathcal{L}_3 : \quad ax + by - a(a-b) - 2bR = 0.$$

It is perpendicular to $\mathcal{L}_2$ at

$$P_{2,3} = \left(\frac{a(a-b)^2 + 2abR}{c^2}, \frac{a(a^2-b^2)+2b^2R}{c^2}\right).$$

Figure 4 also shows an isosceles triangle $PP_{1,x}P_{3,x}$, bounded by the lines $\mathcal{L}_1$, $\mathcal{L}_3$, and OA, with

$$P_{1,x}P_{3,x} = \frac{2bR}{a}, \qquad PP_{1,x} = PP_{3,x} = \frac{cR}{a}.$$

Here, $\dfrac{P_{1,x}T_{3,x}}{P_{1,x}P} = \dfrac{b}{c}$. The inradius of the triangle $PP_{1,x}P_{3,x}$ equals

$$r_3 = T_{3,x}K_3 = R \cdot \frac{b}{c+b} = \frac{bR}{c+b} \tag{1}$$

by the angle bisector theorem. The incenter is the point

$$K_3 = \left(\frac{a(a-b)+bR}{a}, \frac{bR}{c+b}\right).$$

From these the points of tangency can be determined.

The three lines $\mathcal{L}_1$, $\mathcal{L}_2$, and $\mathcal{L}_3$ bound a right triangle $BPP_{2,3}$ with

$$BP_{2,3} = \frac{2ab(a-R)}{ca}, \qquad PP_{2,3} = \frac{(a^2-b^2)(a-R)}{ca}, \qquad BP = \frac{c^2(a-R)}{ca}.$$

This is the right triangle with sides $2ab$, a^2-b^2, $a^2+b^2 = c^2$, magnified by a factor $\dfrac{a-R}{ca} = \dfrac{1}{c+a-b}$. By Proposition 1, it has inradius

$$\begin{aligned} r_4 &= \frac{BP_{2,3}+PP_{2,3}-BP}{2} = \frac{1}{c+a-b} \cdot \frac{2ab+(a^2-b^2)-c^2}{2} \\ &= \frac{1}{c+a-b} \cdot b(a-b) = \frac{b(a-b)}{ca}(a-R) = \frac{bR}{a}. \end{aligned} \tag{2}$$

The incenter K_4 is the intersection of the bisectors of angles $P_{1,x}PP_{2,x}$ and $P_{1,x}BP_{2,y}$:

$$K_4 = \left(\frac{a(a-b)+bR}{a}, \frac{a(a-b)+bR}{a}\right).$$

Proposition 2.

$$r_1 : r_3 : r_4 = \frac{1}{b} : \frac{1}{c+b} : \frac{1}{a}.$$

Proof. From (1) and (2),

$$r_1 : r_3 : r_4 = R : \frac{b}{c+b}R : \frac{b}{a}R = \frac{1}{b} : \frac{1}{c+b} : \frac{1}{a}.$$

□

4. The circles $\mathcal{C}_5$ and $\mathcal{C}_6$

There is a second external common tangent $\mathcal{L}_4$ of $\mathcal{C}_2$ and $\mathcal{C}_4$, which is the reflection of $\mathcal{L}_3$ in the line K_2K_4 (see Figure 5). The line K_2K_4 has an equation

$$ax + ay - 2a(a-b) - 2bR = 0;$$

it intersects $\mathcal{L}_3$ at

$$\left(\frac{a(a-2b)+2bR}{a}, a\right)$$

on the line BC. Since the reflection of $P_{3,x}$ in the line K_2K_4 is the point

$$\left(\frac{2(a(a-b)+bR)}{a}, a-b\right),$$

the second tangent $\mathcal{L}_4$ is the line joining these two points:

$$\mathcal{L}_4 : \qquad abx + a^2y - a(a-b)(a+2b) - 2b^2R = 0.$$

The circle $\mathcal{C}_4$ is tangent to the lines $\mathcal{L}_1$ and $\mathcal{L}_4$ at

$$T_{4,1} = \left(\frac{a(a-b)(a^2+ab+b^2)+b^2(a+b)R}{ac^2}, \frac{a^2(a-b)+b(a+b)R}{c^2} \right),$$
$$T_{4,4} = \left(\frac{a(a-b)(a^2+2b^2)+b(a^2-ab+2b^2)R}{ac^2}, \frac{a(a^3-b^3)+b^2(a+b)R}{ac^2} \right),$$

respectively. We make use of these to construct two circles $\mathcal{C}_5$ and $\mathcal{C}_6$ specified in the Introduction, each congruent, and determine the condition under which $\mathcal{C}_6$ is also tangent to AB. The center of the circle $\mathcal{C}_5$ is

$$K_5 = 2P_{1,4} - K_4 = \left(\frac{a(a-b)(a^2+2ab+3b^2)-b(a^2-3b^2)R}{ac^2}, \frac{a(a-b)(a^2+2ab-b^2)-b(a^2-4ab+b^2)R}{ac^2} \right).$$

The points of tangency are

$$T_{5,1} = 2P_{1,4} - T_{4,1} = \left(\frac{(a-b)(a^2+ab+3b^2)-b^2(a-3b)R}{ac^2}, \frac{a(a-b)(a+2b)-(a-3b)bR}{c^2} \right),$$
$$T_{5,4} = 2P_{1,4} - T_{4,4} = \left(\frac{a(a-b)(a^2+2ab+2b^2)-b(a+b)(a-2b)R}{ac^2}, \frac{a(a-b)(a^2+ab-b^2)+b^2(3a-b)R}{ac^2} \right).$$

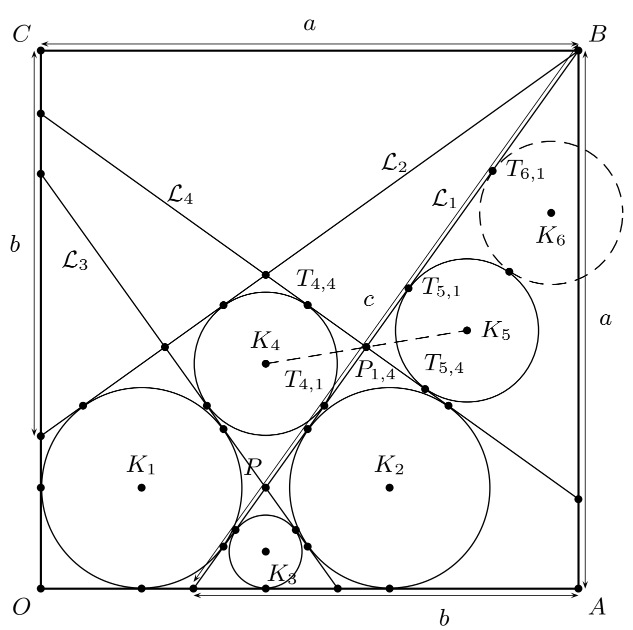

FIGURE 5.

Then we proceed to construct the circle $\mathcal{C}_6$ tangent to $\mathcal{C}_5$ and $\mathcal{L}_1$.

$$\begin{aligned}K_6 &= K_5 + 2(P_{1,4} - T_{4,1}) = (2P_{1,4} - K_4) + 2(P_{1,4} - T_{4,1}) = 4P_{1,4} - 2T_{4,1} - K_4\\ &= \left(\frac{a(a-b)(a^2+2ab+5b^2) - b(a^2+2ab-5b^2)R}{ac^2},\right.\\ &\qquad\left.\frac{a(a-b)(a^2+4ab-b^2) - b(3a^2-6ab+b^2)R}{ac^2}\right),\\ T_{6,1} &= P_{1,4} + 3(P_{1,4} - T_{4,1}) = 4P_{1,4} - 3T_{4,1}\\ &= \left(\frac{a(a-b)(a^2+ab+5b^2) - b^2(3a-5b)R}{ac^2},\right.\\ &\qquad\left.\frac{a(a-b)(a+4b) - b(3a-5b)R}{c^2}\right).\end{aligned}$$

The circles $\mathcal{C}_5$ and $\mathcal{C}_6$ are tangent at

$$\begin{aligned}T'_{5,6} = \frac{K_5+K_6}{2} = &\left(\frac{a(a-b)(a^2+2ab+4b^2) - b(a^2+ab-4b^2)R}{ac^2},\right.\\ &\qquad\left.\frac{a(a-b)(a^2+3ab-b^2) - b(2a^2-5ab+b^2)R}{ac^2}\right).\end{aligned}$$

In general, the circle $\mathcal{C}_6$ is not tangent to AB, as Figure 5 shows.

Proposition 3. *The circle $\mathcal{C}_6$ is tangent to AB if and only if $a : b = 4 : 3$.*

Proof. The circle $\mathcal{C}_6$ is tangent to AB if and only if the difference between a and the x-coordinate of K_6 is equal to r_4. The difference between a and the x-coordinate of K_6 is

$$a - \left(\frac{a(a-b)(a^2+2ab+5b^2) - b(a^2+2ab-5b^2)R}{ac^2}\right) = \frac{b(5b^2-2ab-a^2)}{ac^2}(a-R),$$

and

$$r_4 = \frac{b(a-b)}{ca}(a-R)$$

by (2). Setting these two equal we obtain

$$\frac{5b^2-2ab-a^2}{c} = a-b.$$

Simplifying, we have

$$2b(3a-4b)(a^2-3b^2) = 0.$$

From this, $a : b = 4 : 3$ or $\sqrt{3} : 1$.

(a) For $a : b = 4 : 3$, $R = \frac{3}{8}a$ and

$$r_1 = r_2 = \frac{3}{8}a, \qquad r_3 = \frac{1}{16}a, \qquad r_4 = r_5 = r_6 = \frac{1}{8}a.$$

The circle $\mathcal{C}_6$ is indeed tangent to AB, as shown in Figure 6.

(b) When $a : b = \sqrt{3} : 1$, $R = (2-\sqrt{3})a$ and

$$r_1 = r_2 = (2-\sqrt{3})a, \qquad r_3 = \frac{1}{3}(2-\sqrt{3})a, \qquad r_4 = \frac{1}{3}(2\sqrt{3}-3)a.$$

The centers of $\mathcal{C}_2$ and $\mathcal{C}_5$ are the points

$$K_2 = \left(\frac{(5\sqrt{3}-6)a}{3}, (2-\sqrt{3})a\right), \ K_5 = \left(a, \frac{1}{3}(9-4\sqrt{3})a\right).$$

Since the center K_5 lies on AB, the circle $\mathcal{C}_5$ cannot be completely inside the square. The same is true for $\mathcal{C}_2$ since the x-coordinate of K_2 exceeds $a - R = (\sqrt{3}-1)a$ (see Figure 7). □

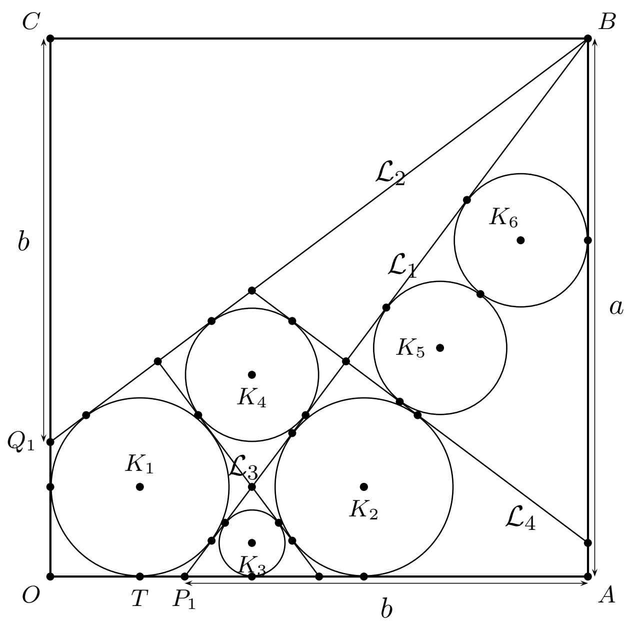

FIGURE 6. $a : b = 4 : 3$

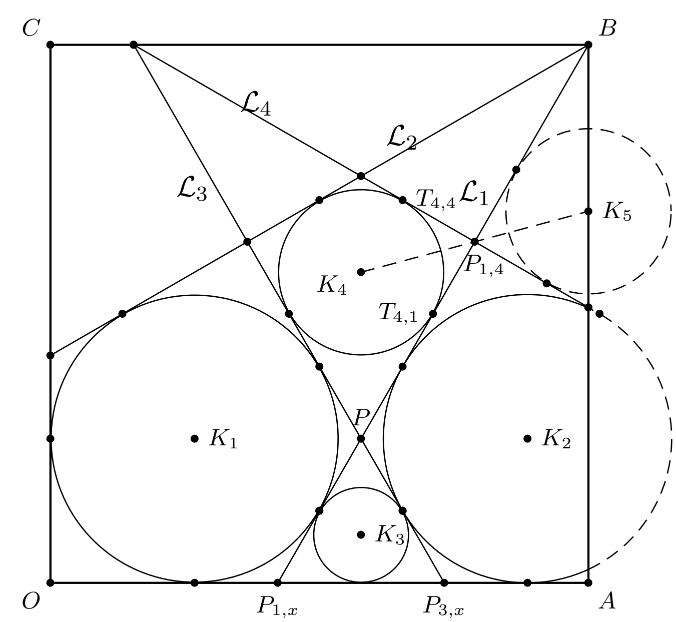

FIGURE 7. $a : b = \sqrt{3} : 1$

5. Appendix

We consider the case when $\mathcal{C}_2$ lies inside the square. This is the case when the x-coordinate of K_2 is less than $a-R$:

$$a - R - \frac{2a(a-b)-(a-2b)R}{a} > 0.$$

Replacing R by $\frac{a(a-b)}{c+a-b}$, we obtain $(2b-a)c - a(a-b) > 0$. Hence

$$2b^3 - 2ab^2 + 2a^2b - a^3 > 0.$$

For a fixed $a > 0$, $f(b) = 2b^3 - 2ab^2 + 2a^2b - a^3$ is an increasing function of b since $f'(b) = 6b^2 - 4ab + 2a^2 > 0$. The only real root of $f(b) = 0$ is $b \approx 0.6478a$ (see Figure 8).

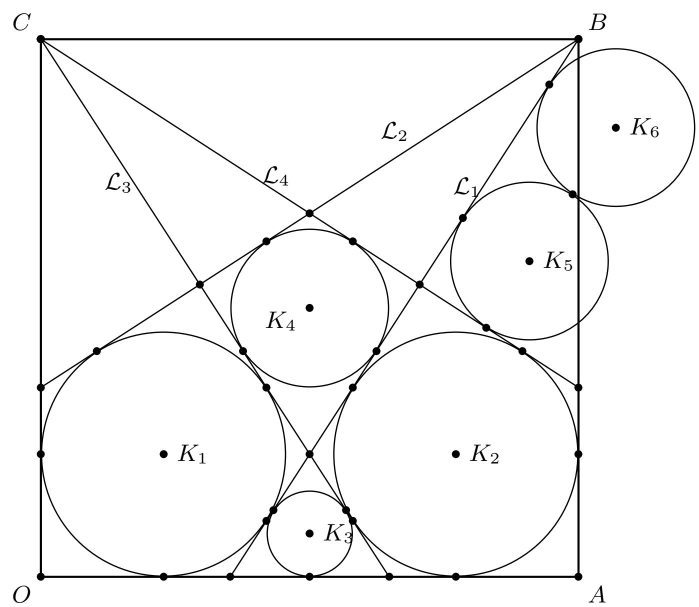

Figure 8. $\mathcal{C}_2$ just fits in the square

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