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The arbelos in Wasan geometry: Ootoba's problem and Archimedean circles

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Abstract. We generalize the problem proposed by Ootoba in the sangaku hung at Takenobu Inari Shrine in Kyoto, and give infinitely many Archimedean circles of the arbelos.

Keywords. arbelos, Archimedean circle

Mathematics Subject Classification (2010). 01A27, 51M04

1. Introduction

Let α, β and γ be the semicircles of diameters AO, BO and AB, respectively for a point O on the segment AB constructed on the same side of AB, where $|AO| = 2a$ and $|BO| = 2b$. The area surrounded by the three semicircles is called an arbelos, and the perpendicular to AB at O is called the axis. The axis divides the arbelos into two curvilinear triangles with congruent incircles, which are called the twin circles of Archimedes and have radius $r_{\mathrm{A}} = ab/(a + b)$. Circles of the same radius are said to be Archimedean (see Figure 1) [2].

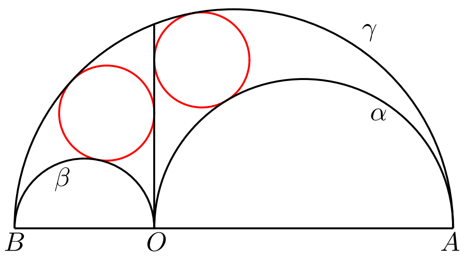

Figure 1.

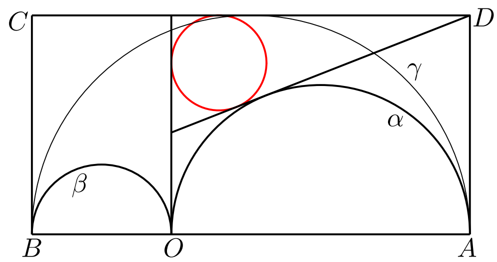

Figure 2.

In this paper, we consider the following problem proposed by Ootoba (大鳥羽源吉守敬), which can be found in the sangaku hung in 1853 at Takenobu Inari Shrine (武信稲荷神社) in Kyoto [6] (see Figure 2).

Problem 1. Let CD be the tangent of γ parallel to AB such that DA and BC are the tangents of γ at the points A and B, respectively. Show that the incircle of the curvilinear triangle made by CD, the remaining tangent of α from D and the axis is Archimedean.

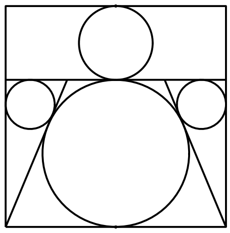

Figure 3.

The original figure of the problem does not explicitly describe an arbelos as shown in Figure 3. There are several problems involving the twin circles of Archimedes in Wasan geometry [1], [3], [4]. On the other hand, it is very rare to find a problem involving an Archimedean circle different from the mate of the twin circles. Furthermore it seems that the Archimedean circle in the problem have not been considered elsewhere until today except for [14], which gives a generalization of the problem. In this paper we give another generalization of the problem, and give infinitely many Archimedean circles.

2. Generalization

We generalize the problem. Let $H \neq A$ be a point lying on the same side of AB as α such that the line AH is the common tangent of α and γ at A and $|AH| = h$ (see Figure 4). We assume that the remaining tangent of α (resp. γ) from H touches α (resp. γ) at a point P (resp. Q), ε_H is the circle of radius e touching α and γ at P and Q, respectively. The incircle of the curvilinear triangle made by the lines HP, HQ and the axis is denoted by δ_H. We use a rectangular coordinate system with origin O such that the farthest point on α from AB has coordinates (a, a). The problem is obtained in the case in which Q coincides with the farthest point on γ from AB in the next theorem:

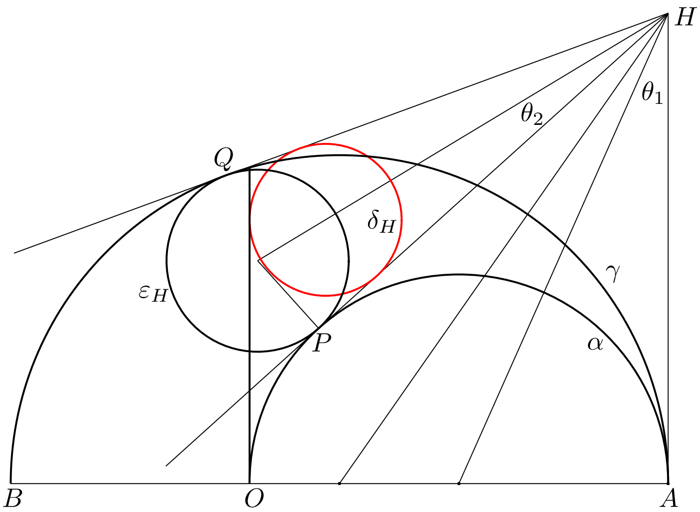

Figure 4.

Theorem 1. *The circle δ_H is Archimedean.*

Proof. We assume that $\angle AHP = 2\theta_1$, $\angle PHQ = 2\theta_2$. Then we have $\tan\theta_1 = a/h$, $\tan\theta_2 = e/h$ and $\tan(\theta_1+\theta_2) = (a+b)/h$. Eliminating θ_1 and θ_2 from the three equations, and solving the resulting equation for e, we get

$$e = \frac{bh^2}{j^2}, \quad where \quad j = \sqrt{a(a+b)+h^2}. \tag{1}$$

Since the point P is the reflection of the point A in the line passing through H and the center of α, it has coordinates

$$(x_p, y_p) = \left(\frac{2a^3}{k^2}, \frac{2a^2h}{k^2}\right), \quad where\ k = \sqrt{a^2+h^2}. \tag{2}$$

Let (x_e, y_e) be the coordinates of the center of ε_H. Then we have $(x_e - x_p)^2 + (y_e - y_p)^2 = e^2$ and $(x_e - 2a)^2 + (y_e - h)^2 = e^2 + h^2$, where notice that if (x_e, y_e) are the coordinates of the reflection of the center of ε_H in the line HP, then they also satisfy the two equations. Solving the two equations for x_e and y_e, we get

$$(x_e, y_e) = \left(\frac{2a^2(a+b) - bh^2}{j^2}, \frac{2a(a+b)h}{j^2}\right) \tag{3}$$

or

$$(x_e, y_e) = \left(\frac{2a^4(a+b) + a^2(2a-b)h^2 + bh^4}{k^2j^2}, 2ah\left(\frac{2a}{k^2} - \frac{a+b}{j^2}\right)\right).$$

While we have

$$\frac{2a(a+b)h}{j^2} - 2ah\left(\frac{2a}{k^2} - \frac{a+b}{j^2}\right) = \frac{4abh^3}{k^2j^2} > 0.$$

Therefore we get (3). If r is the radius of δ_H, then $e/(2a - x_e) = r/(2a - r)$ holds and the last equation implies $r = r_{\mathrm{A}}$ by (1) and (3). □

Notice that k is the distance between the point H and the center of α. Since there are infinite many choices of H, we get infinitely many Archimedean circles touching the axis from the side opposite to the point B. Similar infinitely many Archimedean circles are also obtained from points on the common tangent of β and γ at B.

Proposition 1. *The line PQ passes though a fixed point on AB.*

Proof. The point Q is the reflection of the point A in the line passing through H and the center of γ. Therefore it has coordinates

$$\left(\frac{2a(a+b)^2 - 2bh^2}{l^2}, \frac{2(a+b)^2h}{l^2}\right), \quad where \quad l = \sqrt{(a+b)^2+h^2}. \tag{4}$$

Therefore by (2) and (4), we see that the line PQ passes through the point of coordinates $(2a^2/(2a+b), 0)$. □

Notice that l is the distance between H and the center of γ.

3. Points determining the same Archimedean circle

For the Archimedean circle δ_H, there are two tangents of α (also γ) and δ_H in general, which implies that there is a point $H' \neq H$ on the line AH such that the circle $\delta_{H'}$ coincides with δ_H. In this section we consider a relationship between such two points determining the same Archimedean circles.

The tangent of α from B also touches the Archimedean incircle of the curvilinear triangle made by α, γ and the axis [2]. Hence this circle coincides with the circle δ_H if H lies on this tangent (see Figure 5). In this case we explicitly denote the point H by H_0. Since the point of tangency of α and the tangent has coordinates $(2ab/t, 2a\sqrt{(a+b)b}/t)$, where $t = a + 2b$ [9], the point H_0 has y-coordinate

$$h_0 = a\sqrt{\frac{a+b}{b}}. \tag{5}$$

The circles δ_{H_0} and ε_{H_0} coincide.

Theorem 2. *Let y_d be the y-coordinate of the center of the circle δ_H. Then*

$$y_d = \frac{a^2}{h} + \frac{hr_{\mathrm{A}}}{a} \tag{6}$$

holds, and δ_H is closest to the line AB if and only if $H = H_0$.

Proof. Recall that if two externally touching circles of radii r_1 and r_2 touch a line at two points S and T, then $|ST| = 2\sqrt{r_1 r_2}$ holds. From $e/(h - y_e) = r_{\mathrm{A}}/(h - y_d)$ we get (6) by (1) and (3). Then we get

$$y_d \geq 2\sqrt{\frac{a^2}{h}\frac{hr_{\mathrm{A}}}{a}} = 2\sqrt{ar_{\mathrm{A}}},$$

where notice that $2\sqrt{ar_{\mathrm{A}}}$ equals the y-coordinate of the center of the circle δ_{H_0}. □

The next theorem gives a relationship between two points determining the same Archimedean circle touching the axis (see Figure 6).

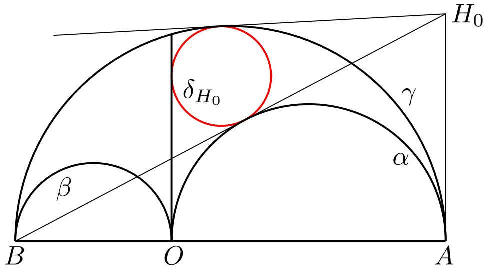

Figure 5.

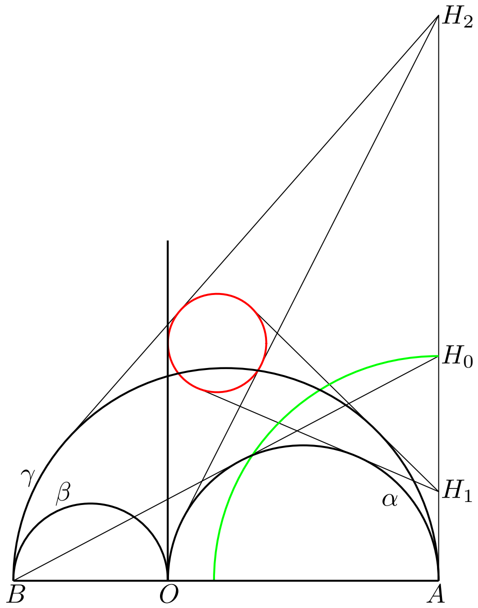

Figure 6.

Theorem 3. *Let H_1 and H_2 be two distinct points different from A on the half line with initial point A passing through H_0. Then the circles δ_{H_1} and δ_{H_2} coincide if and only if the two points are the inverse to each other in the circle of center A and radius h_0.*

Proof. Let $h_i = |AH_i|$. The two circles δ_{H_1} and δ_{H_2} coincide if and only if

$$0 = \left(\frac{a^2}{h_1} + \frac{h_1 r_{\rm A}}{a}\right) - \left(\frac{a^2}{h_2} + \frac{h_2 r_{\rm A}}{a}\right) = \frac{(h_1 - h_2)(bh_1h_2 - a^2(a+b))}{(a+b)h_1h_2}$$

by (6), which is equivalent to $h_1h_2 = h_0^2$ by (5). □

4. Circles of radius b touching the axis

The radius of the excircle of the triangle made by the lines HP, HQ and the axis touching the axis from the side opposite to H equals b by the similarity. We denote this circle by β_H. Notice that H is the external center of similitude of β_H and δ_H. Let y_b be the y-coordinate of the center of β_H. From $b/(h - y_b) = r_{\rm A}/(h - y_d)$ with (6), we have

$$y_b = \frac{a(a+b)}{h}. \tag{7}$$

Let C_1 and C_2 be circles of radii r_1 and r_2, respectively. If d is the distance between their centers, the inclination of the two circles is defined by

$$I(C_1, C_2) = \frac{r_1^2 + r_2^2 - d^2}{2r_1r_2}.$$

The two circles touch externally, are orthogonal, touch internally, according as $I(C_1, C_2) = -1$, $I(C_1, C_2) = 0$, $I(C_1, C_2) = 1$. The next theorem is obtained by (5) and (7) (see Figure 7).

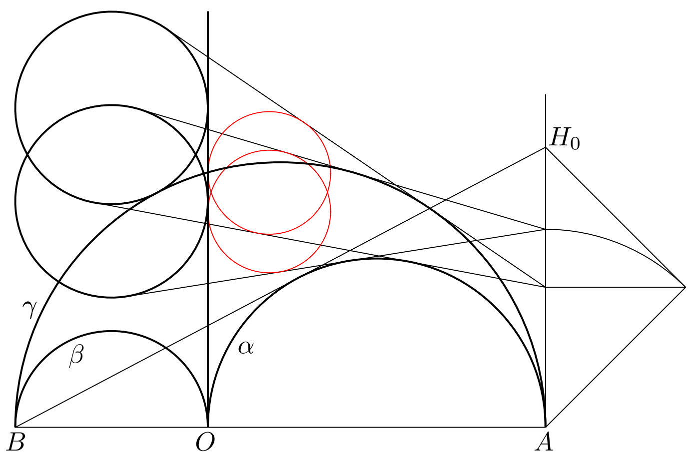

Figure 7.

Theorem 4. *The following statements are true.*

(i) $I(\beta_H, \gamma) = 1 - \dfrac{h_0^2}{2h^2}$.

(ii) *The circles β_H and γ touch externally if and only if $h = h_0/2$.*

(iii) *The circles β_H and γ are orthogonal if and only if $h = h_0/\sqrt{2}$.*

5. The case $H = A$

We consider the case in which the point H coincides with the point A by the definition of division by zero [7]:

$$\frac{z}{0} = 0 \text{ for any real number } z. \quad (8)$$

We now assume that $h = 0$ and δ_H is the Archimedean circle of center (r_A, y_d) satisfying (6). The points P and Q have coordinates $(2a, 0)$ by (2) and (4), i.e., they coincide with the point A. While the circle ε_H has center of coordinates $(2a, 0)$ and radius 0 by (3) and (1). Therefore ε_H also coincides with A. While $y_d = 0$ by (6) and (8). Therefore the circle δ_H touches the axis at the point O (see Figure 8). If H_2 is the inverse of H in the circle of center A and radius h_0, then $h_2 = h_0^2/h$ implies $h_2 = 0$ by (8), where $h_2 = |AH_2|$. Hence H_2 also coincides with A.

If Theorem 4(i) holds in this case, it implies $I(\beta_H, \gamma) = 1$ by (8), i.e., β_H touches γ internally. Therefore β_H touches the axis at O. The same conclusion can also be obtained from (7) by (8). This implies that the external center of similitude of the circles β_H and δ_H coincides with the point H. Notice that all the results in this section are consistent. For more applications of division by zero to Wasan geometry see [5], [8], [10, 11, 12], [13, 15].

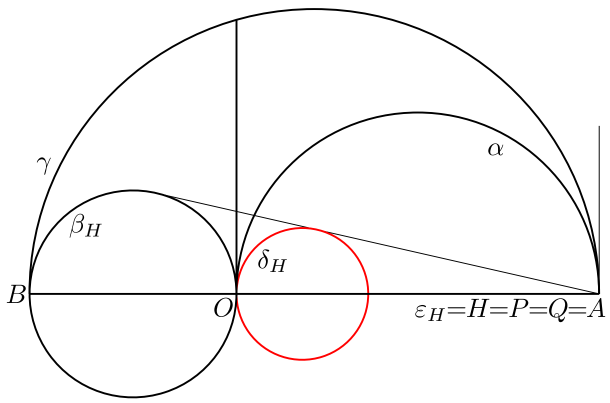

Figure 8: The case $H = A$

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