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Relationships Between Six Circumcircles

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Abstract. If P is a point inside $\triangle ABC$, then the cevians through P divide $\triangle ABC$ into six small triangles. We give theorems about the relationships between the radii of the circumcircles of these triangles. We also state some results about the relationships between the circumcenters of these triangles.

Keywords. Circumcircles, triangle geometry, circumcenters, cevians, cevasix configuration.

Mathematics Subject Classification (2010). 51M04.

1. Introduction

Let P be any point inside a triangle ABC. The cevians through P divide $\triangle ABC$ into six smaller triangles, labeled T_1 through T_6 as shown in Figure 1.

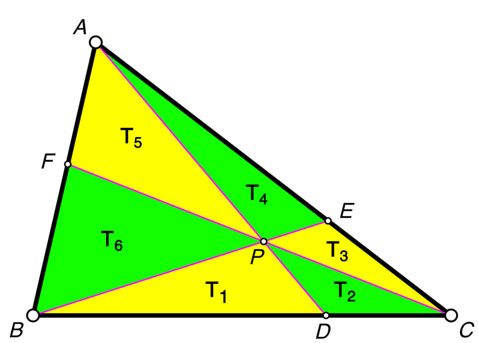

Figure 1. numbering of the six triangles

The relationships between the radii of the circles inscribed in these triangles was investigated in [8]. The relationships between the radii of certain excircles associated with these triangles was investigated in [7]. In this paper, we will investigate the relationships between the radii of the circles circumscribed about these triangles.

We will make use of The Extended Law of Sines which states that if a, b, and c are the lengths of the sides of a triangle opposite angles A, B, and C, then

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

where R is the circumradius of $\triangle ABC$.

2. Radii

We begin with some relationships between the radii of the six circumcircles.

Theorem 2.1. *Let P be any point inside $\triangle ABC$. The cevians through P divide $\triangle ABC$ into six smaller triangles, labeled T_1 through T_6 as shown in Figure 1. Let R_i be the circumradius of T_i. Then $R_1R_3R_5 = R_2R_4R_6$.*

Proof. By The Extended Law of Sines in $\triangle PBD$, we have

$$R_1 = \frac{BD}{2\sin\angle BPD},$$

with similar expressions for the other R_i. Thus,

$$R_1R_3R_5 = \frac{BD}{2\sin\angle BPD}\cdot\frac{CE}{2\sin\angle CPE}\cdot\frac{AF}{2\sin\angle APF}$$

and

$$R_2R_4R_6 = \frac{DC}{2\sin\angle DPC}\cdot\frac{EA}{2\sin\angle EPA}\cdot\frac{FB}{2\sin\angle FPB}.$$

But $BD\cdot CE\cdot AF = DC\cdot EA\cdot FB$ by Ceva's Theorem. Also, angles $\angle BPD$ and $\angle EPA$ are vertical angles, so they are congruent and their sines are equal. Similarly, $\sin\angle CPE = \sin\angle FPB$ and $\sin\angle APF = \sin\angle DPC$. Therefore, we conclude that $R_1R_3R_5 = R_2R_4R_6$. □

Corollary 2.2. *Let P be any point inside $\triangle ABC$. The cevians through P divide $\triangle ABC$ into six smaller triangles, labeled T_1 through T_6 as shown in Figure 1. Let r_i be the radius of the nine-point circle of T_i (the circle through the midpoints of the sides). Then $r_1r_3r_5 = r_2r_4r_6$.*

Proof. This follows immediately from the fact that $R_i = 2r_i$. □

We have some additional results for specific locations of point P.

Theorem 2.3. *Let O be the circumcenter of $\triangle ABC$ and assume that O lies inside $\triangle ABC$. The cevians through O divide $\triangle ABC$ into six smaller triangles, labeled T_1 through T_6 as shown in Figure 1. Let R_i be the circumradius of T_i. Then $R_1 = R_2$, $R_3 = R_4$, and $R_5 = R_6$.*

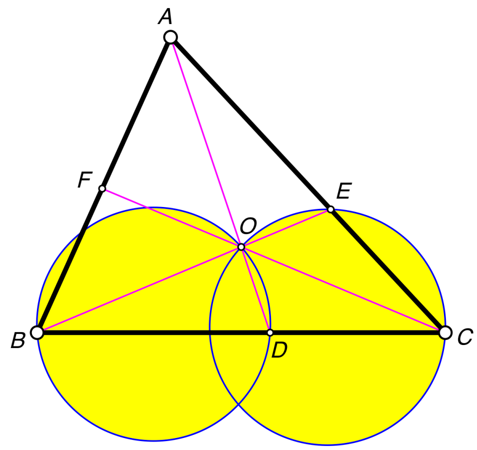

FIGURE 2. Circumcenter: $R_1 = R_2$

Proof. By symmetry, it suffices to show that $R_1 = R_2$ (Figure 2). Since O is the circumcenter of $\triangle ABC$, $OB = OC$. Angles $\angle ODB$ and $\angle ODC$ are supplementary, so their sines are equal. Thus, by The Extended Law of Sines, we have

$$R_1 = \frac{OB}{2\sin\angle ODB} = \frac{OC}{2\sin\angle ODC} = R_2$$

as required. □

Theorem 2.4. *Let N be the Nagel Point of $\triangle ABC$. The cevians through N divide $\triangle ABC$ into six smaller triangles, labeled T_1 through T_6 as shown in Figure 1. Let R_i be the circumradius of T_i. Then $R_1 = R_4$, $R_2 = R_5$, and $R_3 = R_6$.*

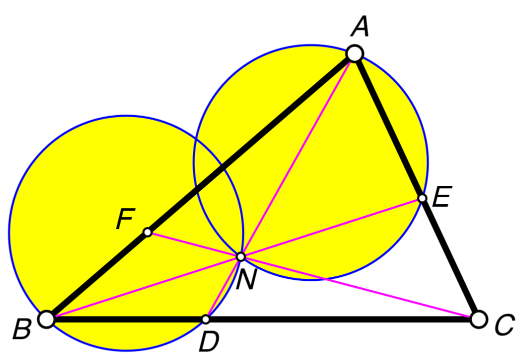

FIGURE 3. Nagel Point: $R_1 = R_4$

Note: The Nagel Point of a triangle is the point of concurrence of AD, BE, and CF, where D, E, and F are the points where the excircles of $\triangle ABC$ touch the sides BC, CA, and AB, respectively [1, p. 160]. The Nagel point is usually denoted Na or N_a, but here we will refer to it as N, for simplicity.

Proof. First note that by symmetry, it suffices to show that $R_1 = R_4$ (Figure 3). If $BC = a$, $CA = b$, $AB = c$, and $s = (a + b + c)/2$, then it is known that

$BD = AE = s - c$ [1, p. 88]. Thus, by The Extended Law of Sines and the fact that $\angle BND = \angle ENA$, we have

$$R_1 = \frac{BD}{2\sin\angle BND} = \frac{AE}{2\sin\angle ENA} = R_4$$

as required. □

Theorem 2.5. *Let H be the orthocenter of $\triangle ABC$. The cevians through H divide $\triangle ABC$ into six smaller triangles, labeled T_1 through T_6 as shown in Figure 1. Let C_i be the circumcircle of T_i. Let R_i be the radius of C_i. Then $R_1 = R_6$, $R_2 = R_3$, and $R_4 = R_5$.*

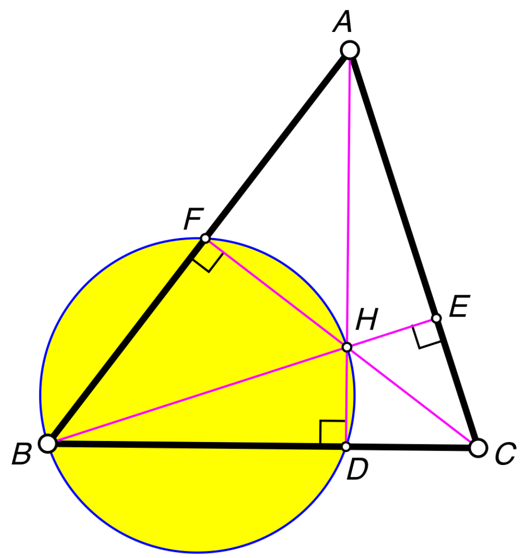

FIGURE 4. Orthocenter: $R_1 = R_6$

Proof. By symmetry, it suffices to show that $R_1 = R_4$ (Figure 4), i.e., that C_1 and C_6 coincide. Since $\angle BDH + \angle HFB = 180°$, quadrilateral $BDHF$ is cyclic. Thus, the circle through points B, D, and H is the same as the circle through points B, F, and H. □

3. CIRCUMCENTERS

Now we collect together some interesting results concerning the centers of the six circumcircles. Most, but not all, of these results are scattered about in the literature. We will use the notation [XYZ] to denote the area of $\triangle XYZ$.

Theorem 3.1. *Let P be any point inside $\triangle ABC$. The cevians through P divide $\triangle ABC$ into six smaller triangles, labeled T_1 through T_6 as shown in Figure 1. Let O_i be the circumcenter of T_i. Then $[O_1O_3O_5] = [O_2O_4O_6]$ (Figure 5).*

The following proof is due to Dubrovsky [2].

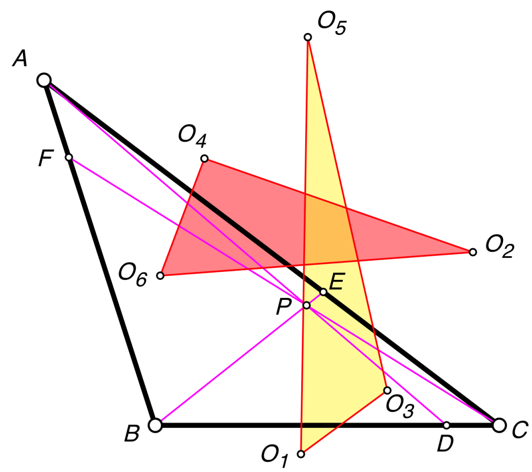

FIGURE 5. Two triangles have same area

Proof. Since O_1 is the circumcenter of $\triangle BPD$, it must lie on the perpendicular bisector of BP. The same remark holds true for O_6. Therefore, $O_1O_6 \perp BP$. In the same way, $O_6O_5 \perp PF$, $O_5O_4 \perp AP$, $O_4O_3 \perp PE$, $O_3O_2 \perp CP$, and $O_2O_1 \perp PD$. Hence $O_1O_6 \parallel O_3O_4$, $O_6O_5 \parallel O_2O_3$, and $O_5O_4 \parallel O_1O_2$. Therefore, hexagon $O_1O_2O_3O_4O_5O_6$ has its opposite sides parallel. But it is known [3] that if $ABCDEF$ is a hexagon with its opposite sides parallel, then $[ACE] = [BDF]$. Thus $[O_1O_3O_5] = [O_2O_4O_6]$. □

Theorem 3.2. *Let M be the centroid of $\triangle ABC$. The medians through M divide $\triangle ABC$ into six smaller triangles, labeled T_1 through T_6 as shown in Figure 1. Let O_i be the circumcenter of T_i. Then $O_1O_4 = O_2O_5 = O_3O_6$. (Figure 6).*

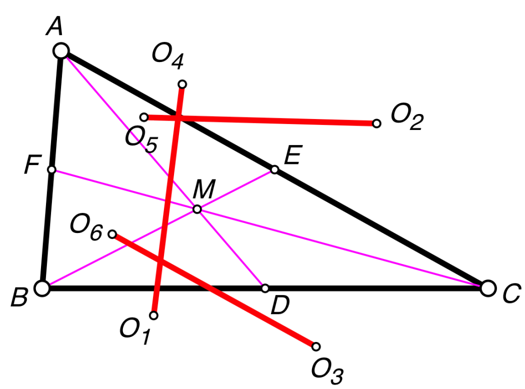

FIGURE 6. Red segments are congruent

Proof. This follows from Proposition 4 of [5]. □

The following result comes from [5].

Theorem 3.3. *Let P be any point inside $\triangle ABC$. The cevians through P divide $\triangle ABC$ into six smaller triangles, labeled T_1 through T_6 as shown in Figure 1. Let*

O_i be the circumcenter of T_i. Then the points O_i lie on a circle if and only if either P is the centroid of $\triangle ABC$ (Figure 7) or P is the orthocenter of $\triangle ABC$ (in which case $O_6 = O_1$, $O_2 = O_3$, and $O_4 = O_5$).

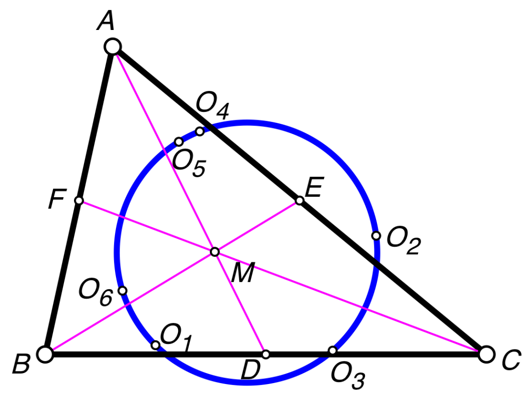

FIGURE 7. O_i lie on a circle when $P = M$

The "if" portion of this theorem is the well-known Van Lamoen's Theorem, [4].

Theorem 3.4. *Let P be any point inside $\triangle ABC$. The cevians through P divide $\triangle ABC$ into six smaller triangles, labeled T_1 through T_6 as shown in Figure 1. Let O_i be the circumcenter of T_i. Then the points O_i lie on a conic (Figure 8).*

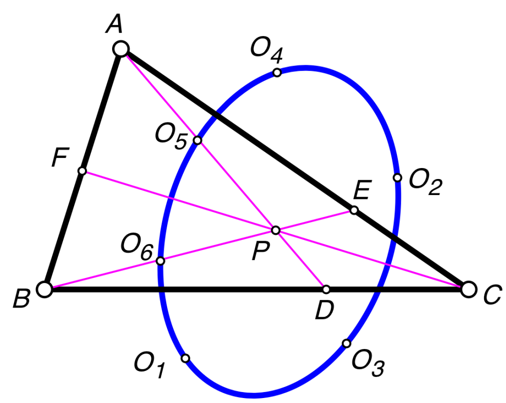

FIGURE 8. O_i lie on a conic

The following proof comes from [5].

Proof. Since $O_1O_6 \parallel O_3O_4$, $O_6O_5 \parallel O_2O_3$, and $O_5O_4 \parallel O_1O_2$, the result follows from the converse of Pascal's Theorem. □

Before stating our next result, we will need the following lemma which comes from [6].

Lemma 3.5. *Let P be any point on the median AD of $\triangle ABC$. Let BE and CF be the cevians through P. Suppose the circumcircles of triangles BPF and CEP meet at points P and Q. Then $\angle BPQ = \angle CPD$ (Figure 9).*

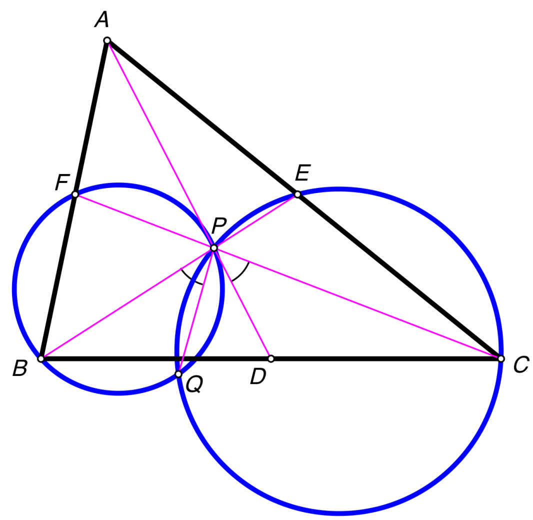

FIGURE 9.

Theorem 3.6. *Let P be any point on the median AD of $\triangle ABC$. The cevians through P divide $\triangle ABC$ into six smaller triangles, labeled T_1 through T_6 as shown in Figure 1. Let O_i be the circumcenter of T_i. Then (Figure 10)*
(a) the points O_1, O_2, O_3, and O_6 lie on a circle,
(b) the points O_3, O_4, O_5, and O_6 lie on a circle.

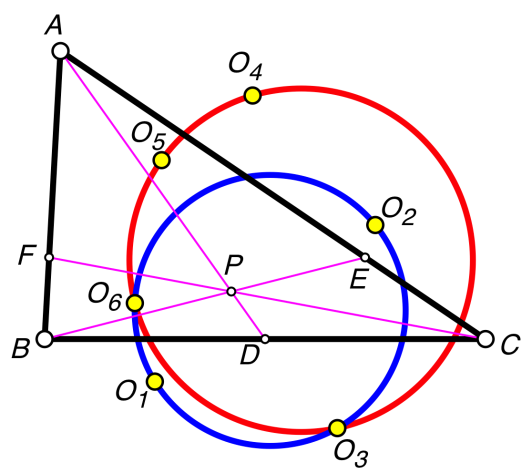

FIGURE 10. The situation when AD is a median

Proof. Part (a). Since O_1 is the circumcenter of $\triangle BPD$, O_1 must lie on the perpendicular bisector of PB. Similarly, O_6 lies on the perpendicular bisector of BP. In the same way, $O_1O_2 \perp PD$ and $O_2O_3 \perp PC$. Draw the circumcircles of triangles PEC and PFB. These circles meet at P and Q (Figure 11). The line joining the centers of two intersecting circles is perpendicular to the common chord, so $O_3O_6 \perp PQ$. Since $\angle PNO_2 = \angle PKO_2$, quadrilateral $PNKO_2$ is cyclic and hence $\angle NPK = \angle NO_2K$. By the same reasoning, quadrilateral $PLMO_6$

is cyclic and so $\angle LPM = \angle LO_6M$. But by Lemma 3.5, $\angle NPK = \angle LPM$. Therefore $\angle NO_2K = \angle LO_6M$. This makes quadrilateral $O_6O_1O_3O_2$ cyclic and O_1, O_2, O_3, and O_6 lie on a circle as claimed. □

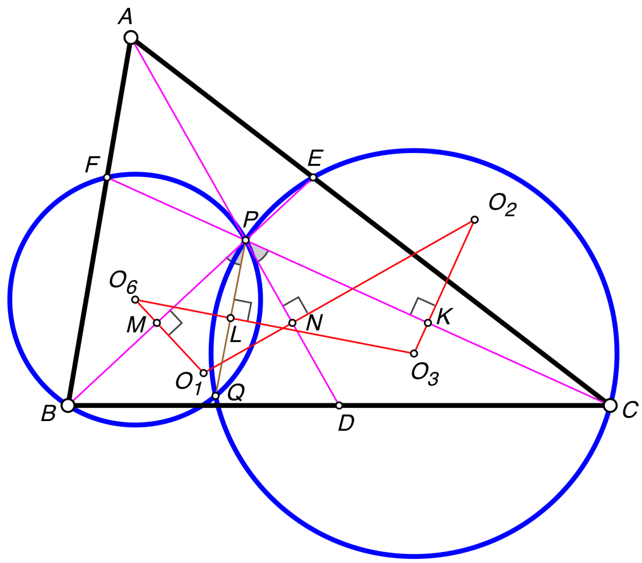

Figure 11.

Part (b). This result is due to Suppa [9]. The proof is similar to the proof of part (a) and details can be found in [6].

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