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Ootoba's Archimedean circles of the collinear arbelos

Hiroshi Okumura
Maebashi Gunma 371-0123, Japan
e-mail: hokmr@yandex.com

Abstract. We generalize Ootoba's Archimedean circle to the collinear arbelos.

Keywords. collinear arbelos, Archimedean circle.

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1. Introduction

For two points P and Q on a line AB in the plane, we denote the semicircle of diameter PQ by (PQ), where all the semicircles with diameters on AB are constructed on the same side. We consider an arbelos formed by the three semicircles (AO), (BO) and (AB) for a point O on the segment AB, where $|AO| = 2a$ and $|BO| = 2b$ (see Figure 1). The perpendicular to AB at the point O is called the axis. Inradius of the curvilinear triangle made by one of (AO) and (BO), (AB) and the axis equals $ab/(a + b)$, and circles of the same radius are said to be Archimedean with respect to the arbelos.

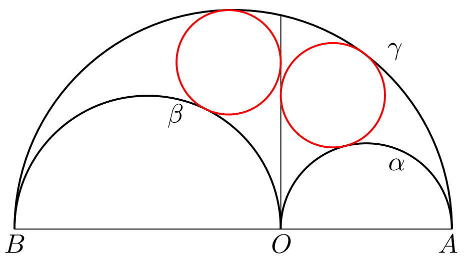

Figure 1.

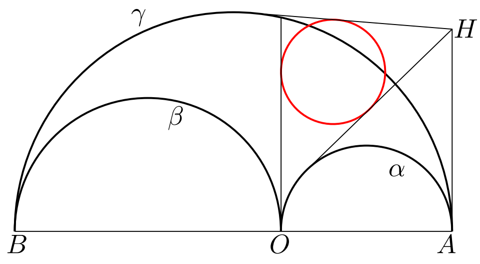

Figure 2.

For a point $H \neq A$ on the common tangent of α and γ at A lying on the same side of AB as γ, we have shown that the incircle of the triangle made by the axis and the remaining tangents of α and γ from H is Archimedean [1, 2] (see Figure 2). The circle is a generalization of the Archimedean circle stated by Ootoba (大鳥羽源吉守敬) in 1853 sangaku hung in Takenobu Inari Shrine (武信稲荷神社)

in Kyoto. In this paper we generalize this circle to a generalized arbelos called a collinear arbelos.

2. Collinear arbelos

For a point P on the half line with initial point A passing through B, let Q be the point on the line AB such that $\overrightarrow{OA} \cdot \overrightarrow{OP} = \overrightarrow{OB} \cdot \overrightarrow{OQ}$, where $\cdot$ is the inner product of the vectors and we assume $P \neq Q$ if $P \neq O$. Let $\alpha = (AP)$, $\beta = (BQ)$ and $\gamma = (AB)$. The configuration consisting of the three semicircles is called a collinear arbelos and denoted by (α, β, γ) [3, 4, 5]. Notice that the axis coincides with the radical axis of the semicircles α and β and the two points P and Q lie between A and B or lie in the order P, B, A, Q (see Figures 3 and 4).

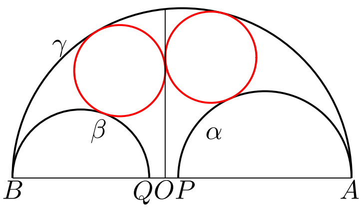

Figure 3.

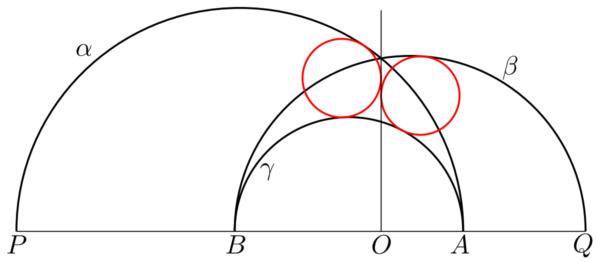

Figure 4.

We use a rectangular coordinate system with origin O such that the points A and B have coordinates $(2a, 0)$ and $(-2b, 0)$, respectively, where we assume that all the semicircles lie on the region $y \geq 0$. Let $(2p, 0)$ and $(2q, 0)$ be the coordinates of the points P and Q, respectively, and let $s = |AQ|/2$ and $t = |BP|/2$. Since $ap + bq = 0$, we have

$$ta = sb \quad and \quad tq + sp = 0. \tag{1}$$

For if P and Q lie between A and B, then $s = a - q$ and $t = b + p$. Therefore $ta - sb = (b + p)a - (a - q)b = ap + bq = 0$. Similarly $tq + sp = 0$. The case, in which P and Q lie in the order P, B, A, Q can be proved in a similar way.

Circles of radius $st/(s+t)$ are called Archimedean circles of (α, β, γ). If $P = O$, then $Q = O$ and (α, β, γ) and its Archimedean circles coincide with the ordinary arbelos mentioned in the opening sentence and its Archimedean circles. Figures 3 and 4 show typical Archimedean circles of (α, β, γ).

3. Generalization

We show that Ootoba's Archimedean circle is generalized to the collinear arbelos. For a point $H \neq A$ on the common tangent of α and γ at A lying on the same side of AB as α for (α, β, γ), let δ_H be the incircle of the triangle made by the axis and the remaining tangents of α and γ from H, if the points P and Q lie between A and B (see Figure 5). If P and Q lie in the order P, B, A, Q, let δ_H be the excircle of the same triangle touching the axis from the side opposite to H (see Figure 6). The next theorem is a generalization of [2, Theorem 1] and [6].

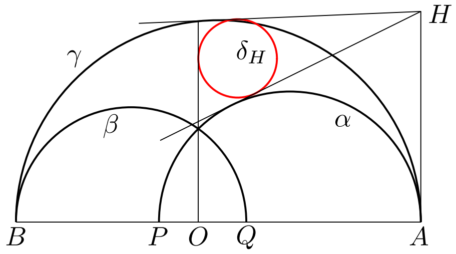

Figure 5.

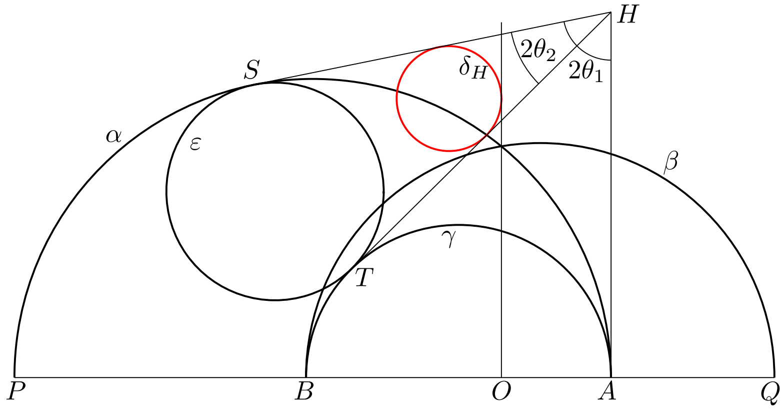

Figure 6.

Theorem 1. *The circle δ_H is an Archimedean circle of (α, β, γ).*

Proof. We prove the case in which P and Q lie in the order P, B, A, Q (see Figure 6). Assume that the tangent of α (resp. γ) from the point H touches α (resp. γ) at a point S (resp. T), $\angle AHS = 2\theta_1$ and $\angle THS = 2\theta_2$. We denote the circle touching α at S and γ at T by ε. Let $|AH| = h$ and let e be the radius of ε. We get $\tan\theta_1 = (a-p)/h$, $\tan\theta_2 = e/h$ and $\tan(\theta_1 - \theta_2) = (a+b)/h$. Eliminating θ_1 and θ_2 from the three equations, and solving the resulting equation for e, we get

$$e = -\frac{h^2(b+p)}{h^2+(a+b)(a-p)}, \tag{2}$$

where notice that $b+p<0$. The line passing through H and the center of α has an equation $y = (x-a-p)h/(a-p)$. Since the point S is the reflection of the point A in this line, it has coordinates

$$(x_s, y_s) = \left(\frac{2(a^3-2a^2p+h^2p+ap^2)}{j^2}, \frac{2h(a-p)^2}{j^2}\right), \tag{3}$$

where $j^2 = (a-p)^2+h^2$. Let (x_e, y_e) be the coordinates of the center of ε. Then we have $(x_s-x_e)^2+(y_s-y_e)^2 = e^2$ and $(x_e-2a)^2+(y_e-h)^2 = e^2+h^2$, where notice that if (x_e, y_e) are the coordinates of the reflection of the center of ε in the line HS, then they also satisfy the same two equations. Substituting (2) and (3) in the two equations and solving the resulting two equations for x_e and y_e, we get

$$(x_e, y_e) = \left(2a-e+\frac{2h^2(-a+e+p)}{j^2}, \frac{2h(a-p)(a-e-p)h}{j^2}\right) \tag{4}$$

or

$$(x_e, y_e) = \left(2a+e-\frac{2h^2(a+e-p)}{j^2}, \frac{2h(a-p)(a+e-p)h}{j^2}\right).$$

However $-e < e$ implies

$$\frac{2h(a-p)(a-e-p)h}{j^2} < \frac{2h(a-p)(a+e-p)h}{j^2}.$$

Therefore we get (4). Let d be the radius of the circle δ_H, then $e/(2a - x_e) = d/(2a + d)$ holds by the similarity. Substituting (2) and (4) in this equation and solving the resulting equation for d, we get

$$d = \frac{-a(b+p)}{a+b}. \tag{5}$$

Then by (1), we get

$$d = \frac{-a(ta/s - tq/s)}{a + ta/s} = \frac{-t(a-q)}{s+t} = \frac{st}{s+t}.$$

Therefore δ_H is an Archimedean circle of (α, β, γ).

The case, where P and Q lie between A and B, is proved similarly, where the equations corresponding to (2), (4) and (5) are obtained as follows, respectively:

$$e = \frac{h^2(b+p)}{h^2 + (a+b)(a-p)}, \tag{6}$$

$$(x_e, y_e) = \left(2a + e - \frac{2h^2(a+e-p)}{j^2}, \frac{2h(a-p)(a+e-p)h}{j^2}\right), \tag{7}$$

$$d = \frac{a(b+p)}{a+b}. \tag{8}$$

□

We still use the notations in the proof. Let y_d be the y-coordinate of the center of the circle δ_H. Since the centers of the circles δ_H and ε and H are collinear, and H does not lie between the two centers, $h - y_d$ and $h - y_e$ have the same sign. Therefore by the similarity, we have $d/(h - y_d) = e/(h - y_e)$. Substituting (2), (4) and (5), or (6), (7) and (8) in this equation and solving the resulting equation for y_d, we get

$$y_d = \frac{a(a-p)}{h} + \frac{bh}{a+b}. \tag{9}$$

If the circles ε and δ_H coincide, then we explicitly denote the point H by H_0. Let $h_0 = |AH_0|$. From $d = e$ and by (2) and (5), or (6) and (8), we get

$$h_0 = \sqrt{\frac{a(a+b)(a-p)}{b}}.$$

Theorem 2. *The circle δ_H is closest to the line AB if and only if $H = H_0$.*

Proof. By (9), we have

$$y_d \geq 2\sqrt{\frac{a(a-p)}{h}\frac{bh}{a+b}} = 2\sqrt{\frac{(a-p)ab}{a+b}},$$

where the equality holds if and only if $a(a-p)/h = bh/(a+b)$. The last equation is equivalent to $h = h_0$. □

Theorem 3. *The Archimedean circles δ_{H_1} and δ_{H_2} coincide if and only if H_2 is the inversion of H_1 in the circle of center A passing through the point H_0.*

Proof. Let $h_i = |AH_i|$. Then $\delta_{H_1} = \delta_{H_2}$ if and only if

$$\frac{a(a-p)}{h_1} + \frac{bh_1}{a+b} = \frac{a(a-p)}{h_2} + \frac{bh_2}{a+b}$$

by (9). This is equivalent to $h_1 = h_2$ or $h_1 h_2 = h_0^2$. □

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