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Solution to Problem 2019-4

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Abstract. Using similar triangles, we solve the problem 2019-4.

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1. Introduction

We solve the following problem (see Figure 1).

Problem. For a square $P_1P_2P_3P_4$, let Q_i $(i = 1, 2, 3, 4)$ be a point on the side P_iP_{i+1} such that $Q_1Q_3 \perp Q_2Q_4$ where the subscripts are taken modulo 4. Let r_i be the radius of the circle lying inside of the square and touching $P_iP_{i\pm1}$ and Q_iQ_{i+2}. Prove or disprove $r_1 + r_3 = r_2 + r_4$.

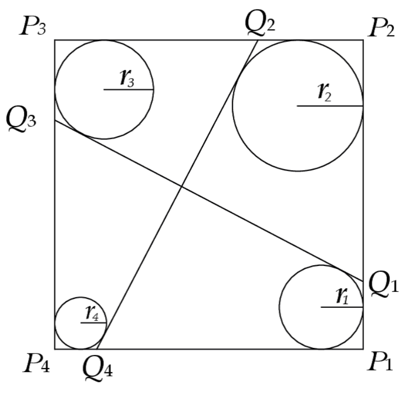

Figure 1.

2. SOLUTION

Without loss of generality, we may assume that the side of the square is the unit. Firstly we consider the special case where Q_1Q_3 and P_1P_4 are parallel (see Figure 2). In this case we obviously have $r_1 + r_3 = 1/2 = r_2 + r_4$.

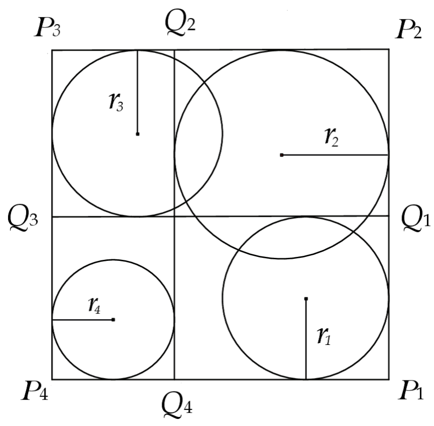

FIGURE 2. Case Q_1Q_3 and P_1P_4 being parallel

Secondly, assume that Q_1Q_3 and P_1P_4 are not parallel. We denote the point of intersection of Q_iQ_{i+2} and P_iP_{i+3} by R_i. Without loss of generality, we may assume that P_1 and R_1 are on the same side of the line P_3P_4. In this case P_i and R_i are on the same side of the line $P_{i+2}P_{i+3}$ where the subscripts are taken modulo 4. If P_1 and R_1 are on not on the same side of the line P_3P_4, just reflect the figure through a line parallel to P_1P_2 and rename several symbols (see Figure 3). If $(Q_1, Q_3) = (P_1, P_3)$, we get $(Q_2, Q_4) = (P_2, P_4)$. Therefore we have $r_1 + r_3 = 0 = r_2 + r_4$.

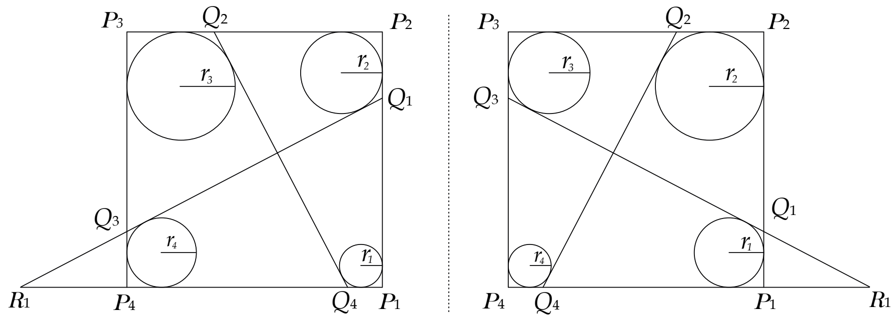

FIGURE 3. Two symmetric configurations

We now assume $Q_1 \neq P_1$. Let $\alpha = P_4Q_3$ and $a = P_4R_1$. Then the inradius of the right-angled triangle $P_4R_1Q_3$ equals $r = \frac{1}{2}\left(a + \alpha - \sqrt{a^2 + \alpha^2}\right)$. By the similarly in Figure 4, we get

$$\frac{r}{a - r} = \frac{r_1}{a - 1 + r_1}.$$

Solving for r_1, we get

(1)

$$r_1 = \frac{(a-1)\,r}{a - 2r} = \frac{a-1}{2}\frac{a + \alpha - \sqrt{a^2 + \alpha^2}}{a - \left(a + \alpha - \sqrt{a^2 + \alpha^2}\right)} = \frac{a-1}{2a}\left(\alpha - a + \sqrt{a^2 + \alpha^2}\right).$$

The triangles $P_4R_1Q_3$ and $P_2R_3Q_1$ are similar. Therefore

$$\frac{r_3}{r_1} = \frac{1-\alpha}{P_1Q_1}.$$

By similar triangles $P_4R_1Q_3$ and $P_1R_1Q_1$ we get

$$\frac{\alpha}{P_1Q_1} = \frac{a}{a-1}.$$

So we deduce that

$$r_3 = \frac{1-\alpha}{2\alpha}\frac{a}{a-1}r_1 = \frac{1-\alpha}{2\alpha}\left(\alpha - a + \sqrt{a^2+\alpha^2}\right).$$

From (1), we get

$$r_1 + r_3 = \frac{a-\alpha}{2a\alpha}\left(\alpha - a + \sqrt{a^2+\alpha^2}\right). \tag{2}$$

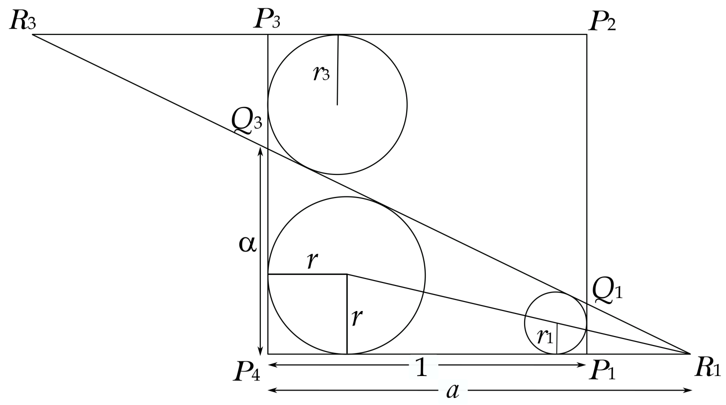

Figure 4.

Let $b = P_1R_2$ and $\beta = P_1Q_4$. Similarly we get

$$r_2 + r_4 = \frac{b-\beta}{2b\beta}\left(\beta - b + \sqrt{b^2+\beta^2}\right).$$

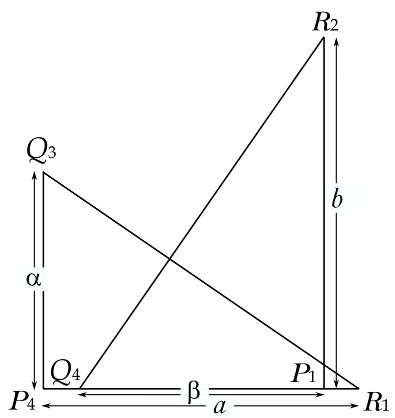

Figure 5.

Since the lines Q_1Q_3 and Q_2Q_4 are perpendicular, the triangles $P_4R_1Q_3$ and $P_1R_2Q_4$ are similar (see Figure 5). So we deduce that

$$\frac{a}{\alpha} = \frac{b}{\beta}.$$

Substituting $a = \alpha b/\beta$ in (2), we get

$$\begin{aligned} r_1 + r_3 &= \frac{\alpha\frac{b}{\beta} - \alpha}{2\alpha\frac{b}{\beta}\alpha}\left(\alpha - \alpha\frac{b}{\beta} + \sqrt{\left(\alpha\frac{b}{\beta}\right)^2 + \alpha^2}\right) \\ &= \frac{b-\beta}{2\alpha b}\left(\alpha - \alpha\frac{b}{\beta} + \frac{\alpha}{\beta}\sqrt{b^2+\beta^2}\right) = r_2 + r_4. \end{aligned}$$

Notice that the above discussion is also true in the case $Q_3 = P_3$.

Assume that Q_1Q_3 and P_1P_4 are not parallel and P_1 and R_1 lie on the same side of P_3P_4. We define $r_1^{(i)}$ as follows: $r_1^{(4)}$ is the inradius of the triangle $P_1Q_1R_1$, $r_1^{(2)}$ is the radius of the excircle of $P_1Q_1R_1$ touching R_1P_1 from the side opposite to Q_1, $r_1^{(3)}$ is the radius of the excircle of $P_1Q_1R_1$ touching P_1Q_1 from the side opposite to R_1, $r_1^{(1)}$ is the radius of the excircle of $P_1Q_1R_1$ touching Q_1R_1 from the side opposite to P_1. Similarly the radii $r_i^{(j)}$ $(i = 2,3,4)$, $(j = 1,2,3,4)$ are defined (see Figure 6). Since the triangles $P_iQ_iR_i$ are all similar, we get the next theorem.

Theorem 1.

$$r_1^{(i)} + r_3^{(i)} = r_2^{(i)} + r_4^{(i)} \quad for \quad i = 1,2,3,4.$$

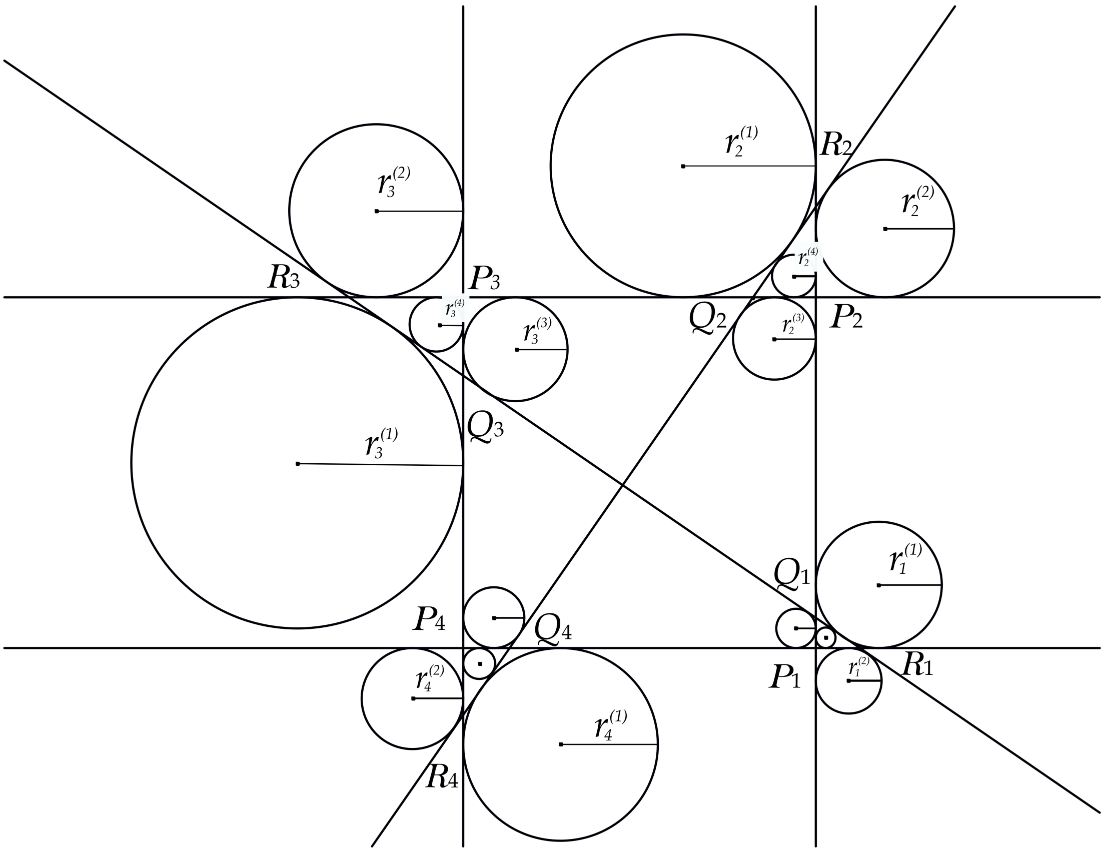

FIGURE 6.