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Solution to 2020-1 Problem 9

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Abstract. An analytic solution to Problem 9 (2020-1) is given.

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1. The problem

We denote the semi-major axis and semi-minor axis of the ellipse by a and b respectively. If r is the radius of the small circle in the Figure 1 , then

$$r^2 - 2r\left(\sqrt{ab} + a + b\right) + ab = 0.$$

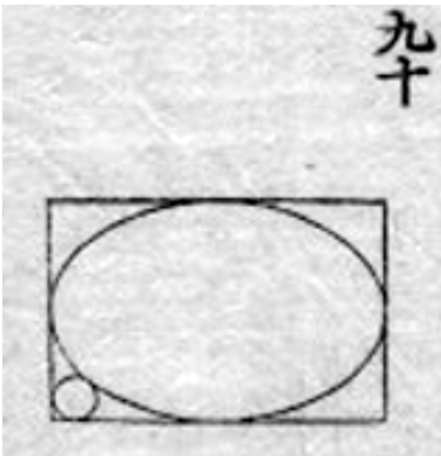

Figure 1.

2. NOTATIONS

We use a rectangular coordinates system with origin O such as the ellipse has equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Let A, B, S be the points with coordinates $(a, 0)$, $(0, b)$ and (a, b) respectively. If M is a point on the ellipse, we denote by $\mathcal{T}_M$ the tangent to the ellipse at the point M. If M is a point on the quarter ellipse that contains A and B and if M is distinct from A and B, we denote the intersection of $\mathcal{T}_M$ and $\mathcal{T}_A$ (respectively $\mathcal{T}_B$) by Q (respectively P) (see Figure 2).

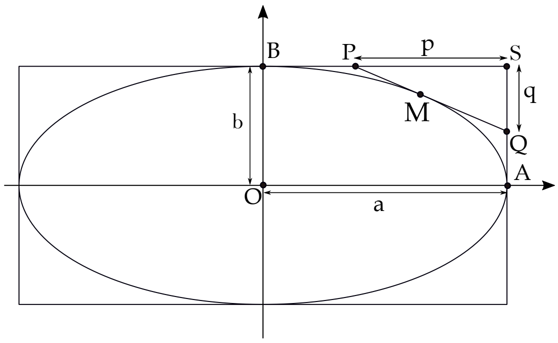

FIGURE 2.

We may assume that P and Q have coordinates

$$P : (a - p, b) \text{ and } Q : (a, b - q)$$

for some reals numbers $p \in]0, a[$ and $q \in]0, b[$.

3. A CONDITION FOR PQ TO BE THE TANGENT TO THE ELLIPSE AT THE POINT M

We look after a condition concerning p and q in order that the line PQ is the tangent $\mathcal{T}_M$ at the ellipse. Let (x_M, y_M) be the coordinates of M, then $\mathcal{T}_M$ has equation

$$\frac{xx_M}{a^2} + \frac{yy_M}{b^2} = 1$$

(cf ref [1]). The points P and Q lie on $\mathcal{T}_M$ if and only if

$$\begin{cases} \frac{(a-p)x_M}{a^2} + \frac{y_M}{b} = 1 \\ \frac{x_M}{a} + \frac{(b-q)y_M}{b^2} = 1. \end{cases}$$

Solving for x_M and y_M, we find

$$x_M = \frac{a^2 q}{ab - (a-p)(b-q)} \text{ and } y_M = \frac{b^2 p}{ab - (a-p)(b-q)}. \tag{1}$$

The condition of tangency is obtained by writing that M is on the ellipse.

$$\frac{x_M^2}{a^2} + \frac{y_M^2}{b^2} = \frac{a^2q^2 + b^2p^2}{(ab - (a-p)(b-q))^2} = 1.$$

Thus we have $a^2q^2 + b^2p^2 = ((a-p)(b-q) - ab)^2$. We expand the right-hand side of this equality, and then we get

$$pq(2ab - 2aq - 2bp + pq) = 0.$$

From the previous formula we deduce the condition of tangency:

$$b - q = \frac{bp}{2a - p}. \tag{2}$$

We now assume that this condition holds. In this case, the lines PQ and $\mathcal{T}_M$ are identical.

4. The incircle of the triangle SPQ

Let $\mathcal{C}$ be the incircle of the triangle SPQ, ρ the inradius and N the point where $\mathcal{C}$ touches the line PQ (see Figure 3).

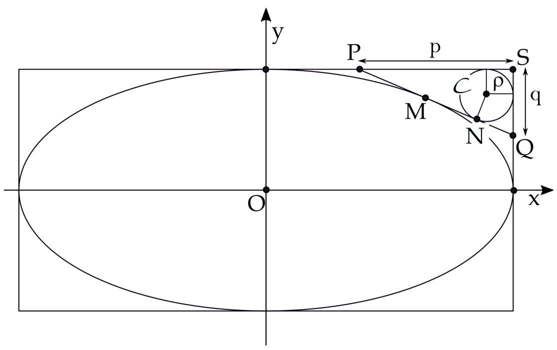

FIGURE 3.

The line PQ has equation

$$q(a - x) + p(b - y) = pq.$$

The vector with coordinates (q, p) is a normal vector to $\mathcal{T}_M$, and the coordinates of the center of $\mathcal{C}$ are $(a - \rho, b - \rho)$. Then the normal line to $\mathcal{C}$ at N has equation

$$p(a - x) - q(b - y) = (p - q)\rho.$$

Because N lies at the intersection of this normal line and the tangent line $\mathcal{T}_M$, the coordinates of N are the solutions of

$$\begin{cases} p(a - x) - q(b - y) = (p - q)\rho \\ q(a - x) + p(b - y) = pq \end{cases}$$

solving for $x - a$ and $y - a$, we get

$$x_N = a - \frac{p}{p^2 + q^2}\left(q(q - \rho) + p\rho\right) \text{ and } y_N = b - \frac{q}{p^2 + q^2}\left(p(p - \rho) + q\rho\right). \tag{3}$$

Since the inradius of the right-angled triangle SPQ is

$$\rho = \frac{p + q - \sqrt{p^2 + q^2}}{2} = \frac{pq}{p + q + \sqrt{p^2 + q^2}},$$

then

$$2\rho = p + q - \sqrt{p^2 + q^2} \tag{4}$$

$$\frac{pq}{\rho} = p + q + \sqrt{p^2 + q^2}. \tag{5}$$

Adding (4) and (5), we get

$$2\rho + \frac{pq}{\rho} = 2(p+q). \tag{6}$$

Solving (6) for q, we get

$$q = 2\rho\frac{p-\rho}{p-2\rho} \text{ and } q-\rho = \frac{p\rho}{p-2\rho}.$$

From the previous equality, we deduce that

$$q(q-\rho) + p\rho = 2\rho\frac{p-\rho}{p-2\rho} \times \frac{p\rho}{p-2\rho} + p\rho = \frac{p\rho\left(p^2 - 2p\rho + 2\rho^2\right)}{(p-2\rho)^2} \tag{7}$$

and with (4), we get

$$\sqrt{p^2+q^2} = p+q-2\rho = p-\rho+\frac{p\rho}{p-2\rho} = \frac{p^2-2p\rho+2\rho^2}{p-2\rho}. \tag{8}$$

Substituting (7) and (8) in (3), we get

$$x_N = a - \frac{p}{\left(\frac{p^2-2p\rho+2\rho^2}{p-2\rho}\right)^2}\left(\frac{p\rho\left(p^2-2p\rho+2\rho^2\right)}{(p-2\rho)^2}\right) = a - \frac{p^2\rho}{p^2-2p\rho+2\rho^2}. \tag{9}$$

and similarly $y_N = b - \frac{q^2\rho}{q^2-2q\rho+2\rho^2}$ (just permute a with b and p with q).

5. The exact value of r

Rewrite x_M as $x_M = a - \frac{ap(b-q)}{ab-(a-p)(b-q)}$, with (2) we get

$$x_M = a - \frac{ap\frac{bp}{2a-p}}{ab-(a-p)\frac{bp}{2a-p}} = a - \frac{p^2a}{p^2-2pa+2a^2}. \tag{10}$$

If the circle $\mathcal{C}$ is the small circle in Figure 1, then $M = N$ and $\rho{=}r$. Using (9) and (10) and replacing ρ by r gives the following equivalence :

$$\begin{aligned} x_M = x_N &\iff a\left(p^2-2pr+2r^2\right) = r\left(p^2-2pa+2a^2\right) \\ &\iff (r-a)\left(2ar-p^2\right) = 0. \end{aligned}$$

Then, because $r \neq a$, we get

$$p^2 = 2ar \text{ and by similarly } q^2 = 2br. \tag{11}$$

We can deduce that

$$r = \frac{p+q-\sqrt{p^2+q^2}}{2} = \frac{\sqrt{2ar}+\sqrt{2br}-\sqrt{2r(a+b)}}{2}.$$

Thus

$$r = \frac{\left(\sqrt{a}+\sqrt{b}-\sqrt{a+b}\right)^2}{2}. \tag{12}$$

Using this value of r to simplify $r^2 - 2r\left(\sqrt{ab}+a+b\right) + ab$ shows that

$$r^2 - 2r\left(\sqrt{ab}+a+b\right) + ab = 0,$$

thus proving Problem 9.

The second solution of $r^2 - 2r\left(\sqrt{ab} + a + b\right) + ab = 0$ is $R = \frac{\left(\sqrt{a}+\sqrt{b}+\sqrt{a+b}\right)^2}{2}$. In section 6, we will find the geometrical significance of R.

6. The second solution

We now assume that the incircle $\mathcal{C}$ of the triangle SPQ is the small circle in Figure 1. In this case with (11) and (12), we get

$$p = \sqrt{2ar} = a - \sqrt{a}\left(\sqrt{a+b} - \sqrt{b}\right),\ q = b - \sqrt{b}\left(\sqrt{a+b} - \sqrt{a}\right) \tag{13}$$

and

$$p^2 - 2ap + 2a^2 = 2a\sqrt{a+b}\left(\sqrt{a+b} - \sqrt{b}\right).$$

Let T be the point where the circle $\mathcal{C}$ touches the ellipse. From theses equalities and (10), we deduce that the coordinates of T are

$$x_T = a - \frac{p^2 a}{p^2 - 2ap + 2a^2} = \frac{2a^2(a-p)}{p^2 - 2ap + 2a^2} = \frac{a\sqrt{a}}{\sqrt{a+b}} \text{ and } y_T = \frac{b\sqrt{b}}{\sqrt{a+b}}. \tag{14}$$

Let Ω be the point with coordinates $(-a+R, -b+R)$. The coordinates of the center C of $\mathcal{C}$ are $(a-r, b-r)$, then from the value of r and R

$$\begin{aligned} r &= a + b + \sqrt{a}\sqrt{b} - \left(\sqrt{a} + \sqrt{b}\right)\sqrt{a+b} \\ R &= a + b + \sqrt{a}\sqrt{b} + \left(\sqrt{a} + \sqrt{b}\right)\sqrt{a+b} \end{aligned}$$

we have

$$\begin{aligned} \Omega T^2 - CT^2 &= (-a + R - x_T)^2 - (a - r - x_T)^2 + (-b + R - y_T)^2 - (b - r - y_T)^2 \\ &= (R + r - 2a)(R - r - 2x_T) + (R + r - 2b)(R - r - 2y_T). \end{aligned}$$

Basic calculations give

$$\begin{aligned} R + r - 2a = 2\sqrt{b}\left(\sqrt{a} + \sqrt{b}\right) \quad &, \quad R + r - 2b = 2\sqrt{a}\left(\sqrt{a} + \sqrt{b}\right) \\ R - r - 2x_T = \tfrac{2\sqrt{b}}{\sqrt{a+b}}\left(b + a + \sqrt{a}\sqrt{b}\right) \quad &, \quad R - r - 2y_T = \tfrac{2\sqrt{a}}{\sqrt{a+b}}\left(b + a + \sqrt{a}\sqrt{b}\right) \end{aligned}$$

thus

$$\Omega T^2 - CT^2 = 4\left(\sqrt{a} + \sqrt{b}\right)\left(b + a + \sqrt{a}\sqrt{b}\right)\sqrt{a+b}$$

and

$$R^2 - r^2 = (R - r)(R + r) = 4\left(\sqrt{a} + \sqrt{b}\right)\left(b + a + \sqrt{a}\sqrt{b}\right)\sqrt{a+b}.$$

The point T lies on the circle $\mathcal{C}$, so $CT^2 = r^2$. We thus deduce that $\Omega T^2 = R^2$ and then the circle $\mathcal{C}'$ with center Ω and radius R is tangent to the ellipse at T. Obviously this circle is also tangent to the lines $x = -a$ and $y = -b$ (see Figure 4).

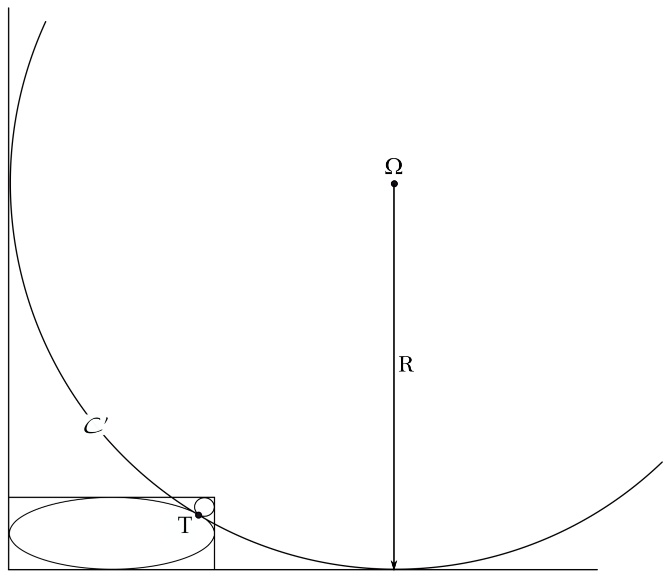

FIGURE 4.

7. An optimization problem

We prove the following theorem.

Theorem 1. *When M lies on the arc $\overset{\frown}{AB}$ of the ellipse, the maximum value of the inradius ρ of the triangle SPQ is r.*

Proof. The condition of tangency (2) shows that $q(p) = b - \frac{bp}{2a-p}$ and $\rho = \frac{1}{2}(p+q-\sqrt{p^2+q^2})$ are continuous functions on $[0,a]$. The extreme value theorem states that ρ obtains a maximum and a minimum value on $[0,a]$. When $p=0$ or $p=a$ we have $\rho = 0$, so the maximum occurs for p in the open set $]0,a[$ and ρ has, at least, one critical point in this interval.

The function $\rho(p)$ is differentiable on $]0,a[$ and

$$\rho' = \frac{1}{2}\left(1 + q' - \frac{p+qq'}{\sqrt{p^2+q^2}}\right).$$

Notice that the condition of tangency (2) is equivalent to

$$(2a-p)(2b-q) = 2ab. \tag{15}$$

Thus if we differentiate (15) with respect to p, we get

$$-(2b-q) - (2a-p)\,q' = 0.$$

Let $u = 2a-p$ and $v = 2b-q$ then $q' = -\frac{v}{u}$ and

$$\rho' = \frac{1}{2u\sqrt{p^2+q^2}}\left(\sqrt{p^2+q^2}\,(u-v) - (pu-vq)\right).$$

Thus if p is a critical number for ρ then

$$\left(p^2+q^2\right)(u-v)^2 - (pu-vq)^2 = 0$$

Expanding the previous equality, and noticing that (15) is equivalent to $uv = 2ab$, we deduce that

$$p^2v^2 + q^2u^2 - 4p^2ab - 4q^2ab + 4pqab = 0.$$

From $p^2v^2 + q^2u^2 = (pv + qu)^2 - 2pquv = (pv + qu)^2 - 4pqab$ we get

$$4ab\left(p^2 + q^2\right) = (pv + qu)^2 .$$

Using $pv + qu = p(2b - q) + q(2a - p) = 2ab - (2a - p)(2b - q) + (2ab - pq)$ and (15) we get

$$4ab\left(p^2 + q^2\right) = (2ab - pq)^2 .$$

Adding $+8abpq$ to both sides of the previous equality, we deduce that

$$4ab\,(p + q)^2 = (2ab + pq)^2$$

and thus, because a, b, p and q are positive,

$$2\sqrt{ab}\,(p + q) = 2ab + pq.$$

The condition of tangency (15) can be rewritten as $2ab + pq = 2aq + 2bp$, then

$$\left(\sqrt{ab} - b\right) p = \left(a - \sqrt{ab}\right) q.$$

Dividing both sides by $\sqrt{a}-\sqrt{b}$, we deduce that $p\sqrt{b} = q\sqrt{a}$. Finally, we substitute $q = \frac{\sqrt{b}}{\sqrt{a}}p$ into (15) and we deduce that

$$p^2 - 2\sqrt{a}\left(\sqrt{a} + \sqrt{b}\right) p + 2a\sqrt{a}\sqrt{b} = 0. \tag{16}$$

Solving for p, we get a unique solution in the open interval $]0, a[$:

$$p = a + \sqrt{a}\sqrt{b} - \sqrt{a}\sqrt{a + b}.$$

Thus ρ has a unique critical point on the open interval $]0, a[$ and ρ attains its maximum value at this point. This value occurs for $p_0 = \sqrt{a}\left(\sqrt{a} + \sqrt{b} - \sqrt{a + b}\right)$.

Let $q_0 = q(p_0) = \dfrac{\sqrt{b}}{\sqrt{a}}p_0$, then the maximum value for ρ is

$$\begin{aligned}\rho\,(p_0) &= \frac{1}{2}\left(p_0 + q_0 - \sqrt{p_0^2 + q_0^2}\right) = \frac{1}{2}\left(p_0 + \frac{\sqrt{b}}{\sqrt{a}}p_0 - \sqrt{p_0^2 + \frac{b}{a}p_0^2}\right)\\ &= \frac{p_0}{2\sqrt{a}}\left(\sqrt{a} + \sqrt{b} - \sqrt{a + b}\right) = \frac{\left(\sqrt{a} + \sqrt{b} - \sqrt{a + b}\right)^2}{2} = r.\end{aligned}$$

This completes the proof. □

Remark. The following results are left as an exercise to the reader :
Let $p_1 = a + \sqrt{a}\sqrt{b} + \sqrt{a}\sqrt{a + b}$ the second solution of (16), $q_1 = \frac{\sqrt{b}}{\sqrt{a}}p_1$ and P_1, Q_1 the points with coordinates $(a - p_1, b)$ and $(a, b - q_1)$ respectively. Then the lines P_1Q_1 is parallel to $\mathcal{T}_T$ and is tangent to the ellipse. The incircle of the triangle SP_1Q_1 is tangent to $\mathcal{T}_T$ and its radius is $\sqrt{ab}$ (see Figure 5).

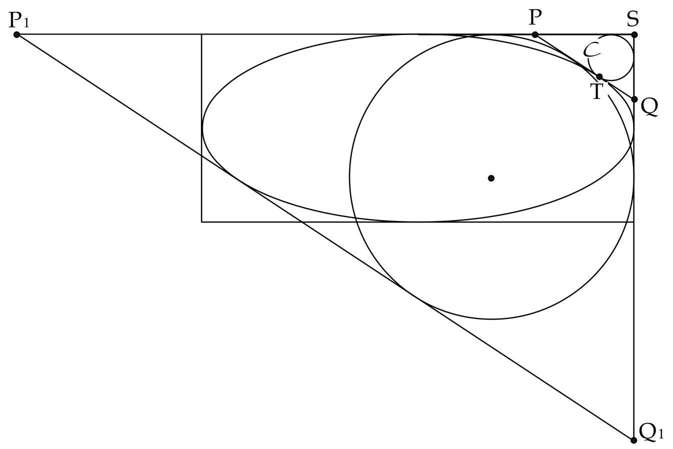

FIGURE 5.

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