Sangaku Journal of Mathematics (SJM) ©SJM
ISSN 2534-9562
Volume 4 (2020), pp.70-72
Received 20 April 2020. Published on-line 30 April 2020
web: http://www.sangaku-journal.eu/

Solution to Problem 2020-2

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Abstract. A geometrical solution to Problem 2020-2 is given.

Keywords. arbelos.

Mathematics Subject Classification (2010). 51M04.

1. Introduction

We will solve the following Problem (see Figure 1).

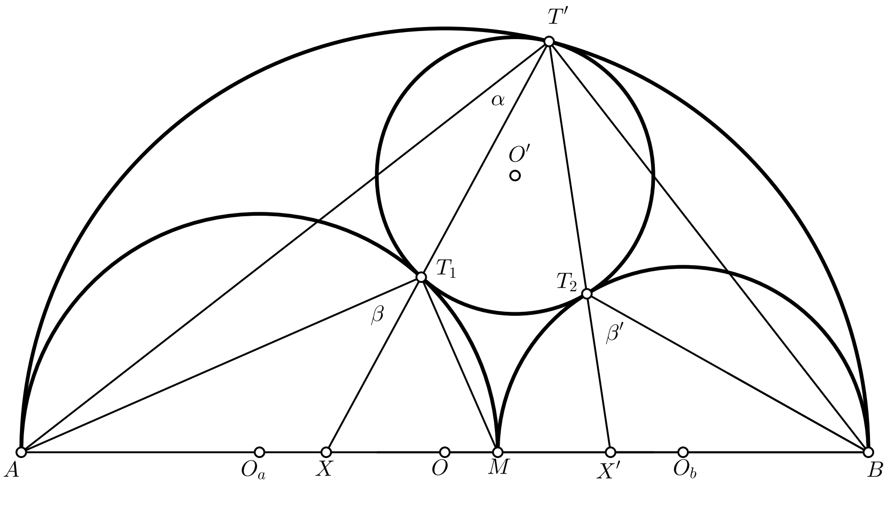

Figure 1.

Problem. For a point M on the segment AB, let (O_a), (O_b) amd (O) be the semicircles of diameters AM, BM and AB, respectively, where the three semicircles form an arbelos and $AM = 2a$ and $BM = 2b$. Assume that the incircle (O') of the arbelos touches (O_a), (O_b) and (O) at points T_1, T_2 and T', respectively, and the lines $T'T_1$ and $T'T_2$ meet AB in the points X and X' respectively. Let

$\angle AT'T_1 = \alpha$, $\angle AT_1X = \beta$ and $\angle BT_2X' = \beta'$. Prove that the following relations hold.

(1) $\beta + \beta' = 90°$,

(2) $\cot\beta = \dfrac{a}{b}$,

(3) $\cot\alpha = \dfrac{a}{b} + 1$.

2. Solution

We will use the following well known lemma (see Figures 2 and 3).

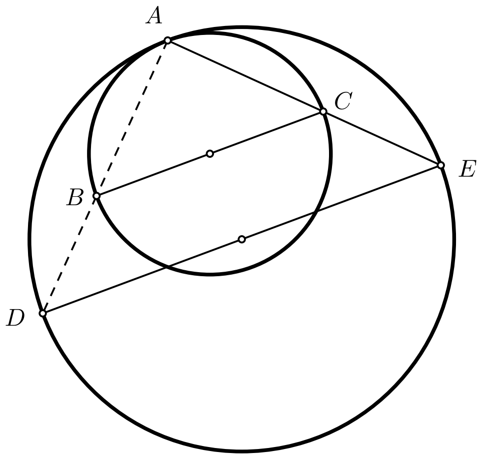

Figure 2. Internal case.

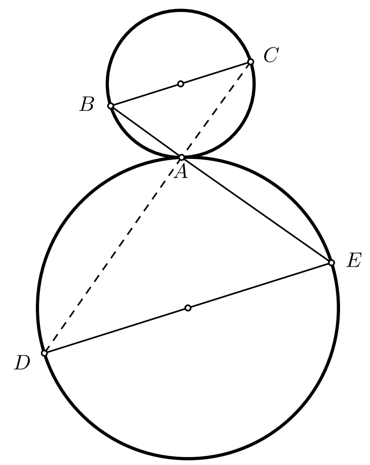

Figure 3. External case.

Lemma 1. *If two circles touch at A (internally or externally), and if BC and DE are parallel diameters, then ACE or ABE is a straight line.*

Now look at the Figure 4. We have, $AM = 2a$, $MB = 2b$ and $AB = 2(a+b)$. Join O', O_a and O', O_b and O', O. Now if r is the radius of (O'), then from the $\triangle O'O_aO_b$ we get, $O'O_a = a+r$, $O'O_b = b+r$, $O_aO_b = a+b$ and $O'O = a+b-r$.

Now applying Stewart's Theorem on $\triangle O'O_aO_b$, we get

$$b(b+r)^2 + a(a+r)^2 = (a+b)[(a+b-r)^2 + ab],$$

which implies

$$r = \frac{ab(a+b)}{a^2+ab+b^2}.$$

Now draw $O'M' \perp O_aO_b$. Let $O'M' = h$. By Pappus' chain theorem, we get $h = 2r$. Let CD be the diameter of the circle (O') parallel to AB.

Now by Lemma 1 we get, ACT' and BDT' are straight line (internal case). Also AT_1D and BT_2C are straight lines (external case). Also for same reasoning MT_1C and MT_2D are straight lines. Let AT' cuts (O_a) at P and BT' cuts (O_b) at Q. Join M, P and M, Q which meet AT_1 at R and BT_2 at S respectively. Let CR and DS produced to meet AB at Y and Y' respectively.

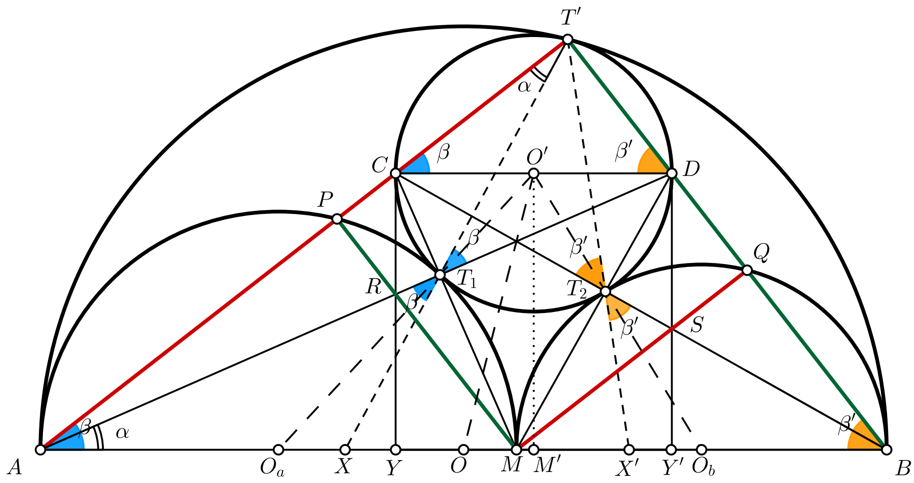

FIGURE 4.

Now we get $\beta = \angle AT_1X = \angle T'T_1D = \angle T'CD$ as they are on the same side of the chord $T'D$ in the circle (O'). Exactly in similar way, we get, $\beta' = \angle BT_2X' = \angle CT_2T' = \angle CDT'$.

Now as in $\triangle T'CD$, $\angle CT'D = 90°$ so we get $\beta + \beta' = 90°$

Now as $CD \parallel AB$, we also get $\angle T'AB = \beta$.
Again $\angle T'AT_1 = \angle AT_1X - \angle AT'T_1 = \beta - \alpha$. So $\angle T_1AX = \alpha$.

In $\triangle AMC$, $MP \perp AC$, $AT_1 \perp MC$, so $CY \perp AM$. By similar reasoning we get $DY' \perp MB$. Now as $\angle APM$, $\angle MQB$ and $\angle AT'B$ all right angles, we get, $MP \parallel BT'$ and $MQ \parallel AT'$ and also we have $CY \parallel DY'$. So we get

$$\frac{AM}{MB} = \frac{AR}{RD} = \frac{AY}{YY'},$$

which implies

$$\frac{AY}{YY'} = \frac{2a}{2b} = \frac{a}{b}.$$

Now as $YY' = CD = 2r$ so we get,

$$AY = 2r \cdot \frac{a}{b}.$$

Now from the $\triangle ADY'$ we get

$$\cot\alpha = \frac{AY'}{DY'} = \frac{2r \cdot \frac{a}{b} + 2r}{2r} = \frac{a}{b} + 1,$$

and from the $\triangle ACY$ we get

$$\cot\beta = \frac{AY}{CY} = \frac{2r \cdot \frac{a}{b}}{2r} = \frac{a}{b}.$$

Editor's comment. A solution using a coordinate system is given by Juan Jose Isach Mayo (Spain).