Sangaku Journal of Mathematics (SJM) ©SJM
ISSN 2534-9562
Volume 4 (2020), pp. 73–76
Received 15 June 2020. Published on-line 18 June 2020
web: http://www.sangaku-journal.eu/

Solution to Problem 2017-1-6 with division by zero

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Abstract. We give a solution of the sixth problem in Problems 2017-1 and show a simple property of the figure with the definition of division by zero.

Keywords. circle chain, division by zero.

Mathematics Subject Classification (2010). 01A27, 51M04.

1. Introduction

In this article we give a solution of the 6th problem in Problems 2017-1 in [15]. We can see the figure of the problem, but there is no text remaining. The figure can be obtained by removing one small circle from the figure in the sangaku problem proposed by Tsunoda (角田義之輔利勝) in 1898 [14]. Let $ABCD$ be a square of side length a. Tsunoda's problem is stated as follows (see Figure 1):

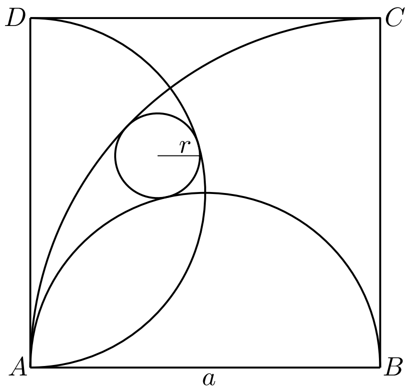

Figure 1.

Problem 1. For the square $ABCD$, let r be the radius of the circle touching the circle of diameter DA internally and the circle of diameter AB externally and the circle of center B passing through A internally. Show $r = 4a/33$.

In this paper we show that the figure can be embedded into a configuration consisting of the square $ABCD$, a chain of circles touching the circle of diameter AB externally and the circle of center B passing through A internally, and circles touching a circle of the chain and the side AB at A. We also show that the last circles have radii forming a harmonic progression.

2. A CHAIN OF CIRCLES

We consider a chain of circles in this section. We define circles and semicircles as follows for the square $ABCD$: α is the semicircle of diameter AB lying inside of $ABCD$, β is the image of α by the homothety with center A and ratio 2, γ_1, γ_2, γ_3, $\cdots$ is a chain of circles touching α and β, where γ_1 is the incircle of the curvilinear triangle made by α, β and the line AB. We use a rectangular coordinate system with origin A so that the point C has coordinates (a, a). The next theorem holds.

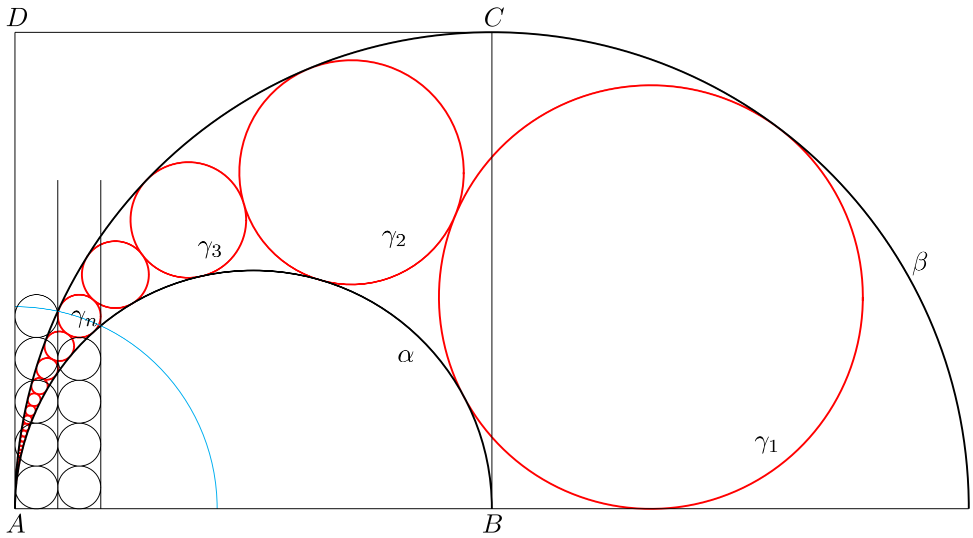

Figure 2: $n = 5$.

Theorem 1. *The circle γ_n has radius $c_n = 4a/(4n^2 - 4n + 9)$ and center of coordinates $(3c_n, (2n-1)c_n)$. Therefore the smallest circle touching γ_n externally and the side DA is congruent to γ_n.*

Proof. Let (p, q) be the coordinates of the center of γ_n. Inverting the circles α, β, γ_1, γ_2, $\cdots$, γ_n by the inversion in the circle of center A orthogonal to γ_n, we get $q = (2n-1)c_n$ (see Figure 2). Then the equations $(p-a)^2 + q^2 = (a - c_n)^2$ and $(p - a/2)^2 + q^2 = (c_n + a/2)^2$ yield $c_n = 4a/(4n^2 - 4n + 9)$ and $p = 3c_n$. □

3. A GENERALIZATION OF THE PROBLEM

In this section we prove the fact stated at the end of section 1, which gives a generalization of Problem 1. Let δ_0 be the line AB and let δ_n be the circle of radius d_n touching AB at A and also touching γ_n externally for $n = 1, 2, 3, \cdots$.

Theorem 2. *The relation $d_n = \dfrac{a}{n}$ holds for $n = 1, 2, 3, \cdots$.*

Proof. By Theorem 1, $(3c_n)^2 + ((2n-1)c_n - d_n)^2 = (c_n + d_n)^2$ holds, which gives the equation of the theorem. □

Inverting the figure in the circle of center A orthogonal to γ_n, we see that the inverse of δ_n, which is a line parallel to AB, touches the inverses of γ_n and γ_{n+1} at their point of tangency. Hence the circle δ_n touches the circles γ_n and γ_{n+1} at their point of tangency. We now see that δ_2 is the circle of diameter DA by Theorem 2. Hence the circle γ_3 coincides with the circle of radius r in Problem 1, i.e., Figure 1 can be embedded into the configuration consisting of $ABCD$, δ_0, the circles α, β, γ_i, δ_i $(i = 1, 2, 3, \cdots)$ (see Figure 3). Theorem 1 is now a generalization of Problem 1.

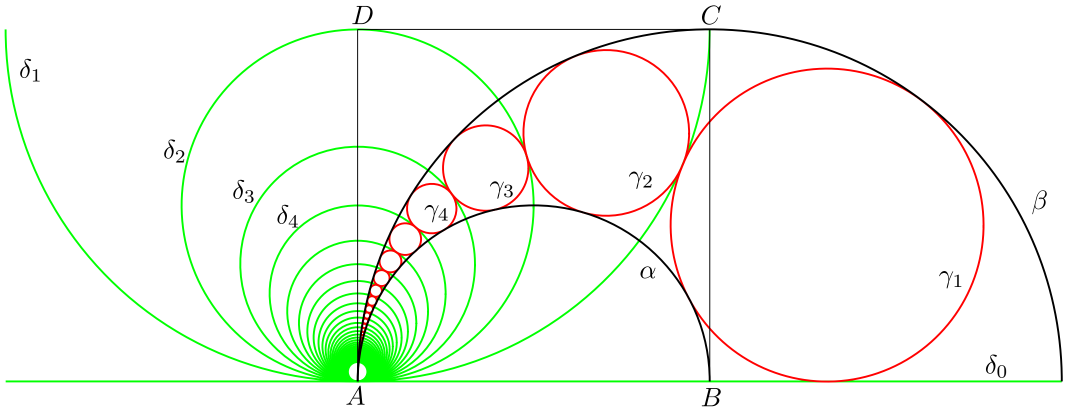

Figure 3.

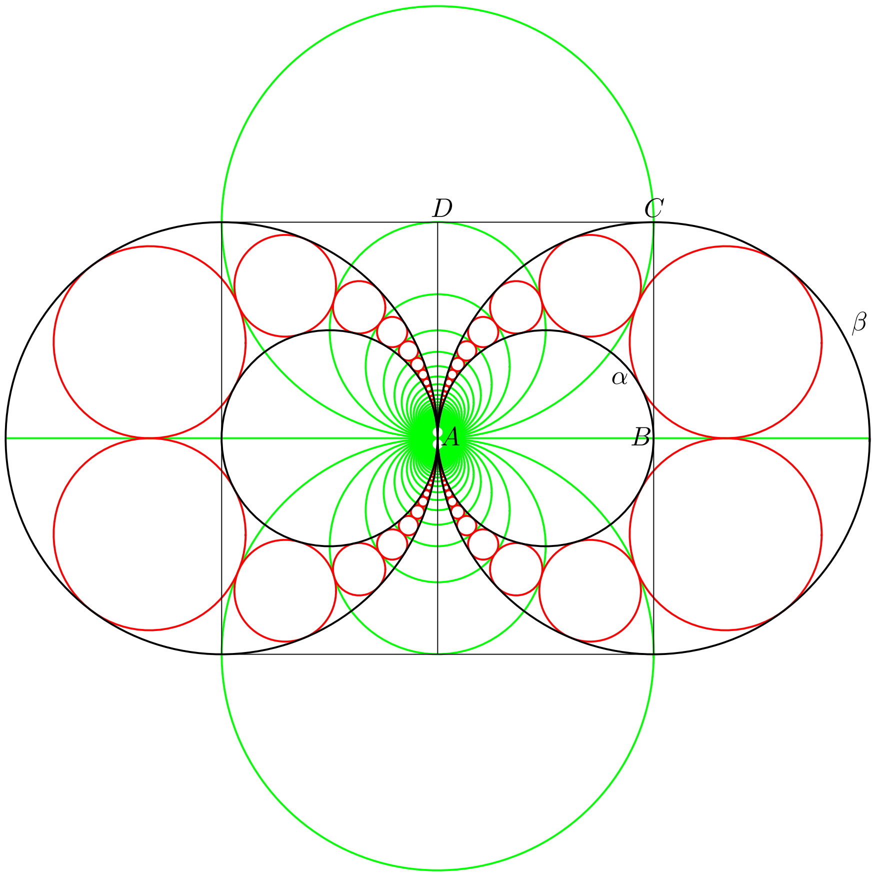

Figure 4.

We now add the reflections of $ABCD$, the semicircles α, β and the circle γ_i $(i = 1, 2, 3, \cdots)$ in the line AB in Figure 3. With the resulting figure and its image by the rotation around A through 180°, we get a a symmetric configuration of circles and squares (see Figure 4). Notice that Figure 1 is a part of this configuration.

4. THE CASE $n = 0$ WITH DIVISION BY ZERO

In this section we consider the case $n = 0$ in Theorem 2 using the definition of division by zero [3]:

$$\frac{z}{0} = 0 \textit{ for any real number } z. \tag{1}$$

Notice that reduction for fractions of zero denominator can not be used with this definition, i.e., in general we have

$$\frac{ac}{bc} \neq \frac{a}{b} \quad if \quad c = 0.$$

A circle or a line has an equation $S(x, y) = e(x^2 + y^2) + 2fx + 2gy + h = 0$. If $S(x, y) = 0$ expresses a circle, its radius is given by $R = \sqrt{(f^2 + g^2 - eh)/e^2}$. Since $e = 0$ implies $R = 0$ by (1), we can conclude that *any line has radius* 0 by (1) [13]. Therefore if d_0 is the radius of the line δ_0, we get $d_0 = 0$. On the other hand, $a/0 = 0$ by (1). Therefore Theorem 2 still holds in the case $n = 0$.

For a brief introduction of division by zero and division by zero calculus see [12], and its application to Wasan geometry see [1], [2], [4, 5, 6, 7, 8, 9, 10], [11]. For an extensive reference of division by zero and division by zero calculus, see [13].

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