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Relationships Between Circles Inscribed in Triangles and Related Curvilinear Triangles

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Abstract. If P is a point inside $\triangle ABC$, then the cevians through P extended to the circumcircle of $\triangle ABC$ create a figure containing a number of curvilinear triangles. Each curvilinear triangle is bounded by an arc of the circumcircle and two line segments lying along the sides or cevians of the original triangle. We give theorems about the relationships between the radii of circles inscribed in various sets of these curvilinear triangles.

Keywords. circles, cevians, curvilinear triangles, sangaku.

Mathematics Subject Classification (2010). 51M04.

1. INTRODUCTION

A *curvilinear triangle* is a geometric figure bounded by three curves. The curves are typically line segments and arcs of circles, in which case there is a unique circle tangent to each of the three boundary curves. This circle is called the *incircle* of the curvilinear triangle.

Wasan geometers loved to find relationships between the radii of circles inscribed in curvilinear triangles. An example is shown in Figure 1 which comes from an 1841 book of Mathematical Formulae written by Yamamoto [12]. It is also given as problem 5.3.9 in [4]. In the figure, $AH \perp BC$ and $BA \perp AC$. There are three curvilinear triangles of interest in the figure. The first curvilinear triangle is bounded by BH , HA , and arc \widehat{AB} . The second curvilinear triangle is bounded by AH , HC , and arc \widehat{CA} . The third curvilinear triangle is bounded by CA , AB ,

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and arc \widehat{BC} . The radii of the circles inscribed in these curvilinear triangles are r_1 , r_2 , and r_3 , respectively. Then the nice relationship that was found is $r_1 + r_2 = r_3$.

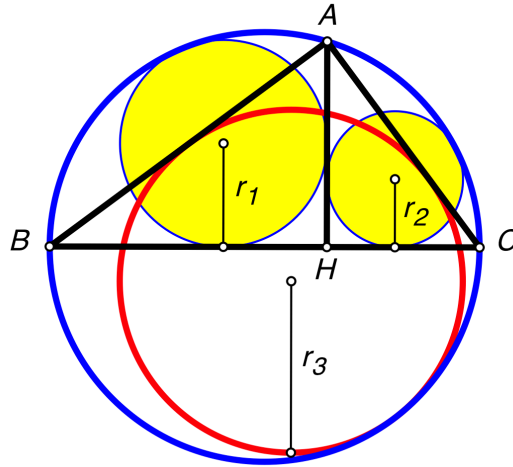


FIGURE 1. $r_1 + r_2 = r_3$

In this paper, we will find some other nice relationships between the inradii of curvilinear triangles.

If a curvilinear triangle is convex and bounded by two straight line segments and one circular arc, then we will call the resulting figure a *skewed sector* (see Figure 2).

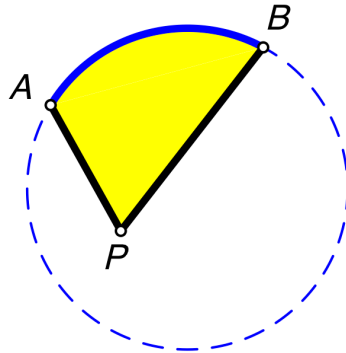


FIGURE 2. a skewed sector

Anatomy of a skewed sector.

- The two straight line segments are called the *sides* of the skewed sector.
- The point of intersection of the two sides is called the *vertex* of the skewed sector.
- The angle between the two sides is called the *vertex angle*.
- The circular arc is referred to as the *arc of the skewed sector*.
- The circular measure of the arc of a skewed sector is called the *arc angle*.

- The circle to which the arc belongs will be called the *circle associated with the skewed sector*.
- The triangle formed by the vertex of a skewed sector and the endpoints of its arc will be referred to as the *triangle associated with the skewed sector*. This would be $\triangle APB$ in Figure 2.
- When naming a skewed sector, the vertex will always be the middle letter. Thus, the skewed sector in Figure 2 is named skewed sector APB .
- When the vertex of a skewed sector lies inside the associated circle, if the sides of the skewed sector are extended back through the vertex, they will intercept an arc of the associated circle. This arc is called the *opposite arc* of the skewed sector. It is shown in red in Figure 3.
- The vertex of a skewed sector and the opposite arc form another skewed sector called the *opposite skewed sector*. This is skewed sector $A'PB'$ in Figure 3.

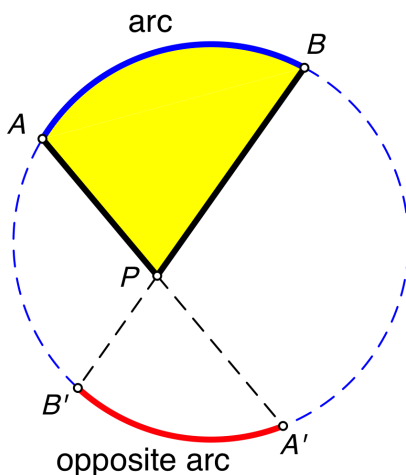


FIGURE 3. opposite arc

A *segment* of a circle is the figure bounded by an arc of a circle and the chord joining the endpoints of that arc. The height (or sagitta) of the segment is the distance from the midpoint of the chord to the midpoint of the arc.

If P is a point inside $\triangle ABC$, then the cevians through P extended to the circumcircle of $\triangle ABC$ create a figure containing a number of skewed sectors. We will find relationships between the radii of the circles inscribed in some of these skewed sectors.

Notation.

- If X and Y are points, then we use the notation XY to denote either the line segment joining X and Y or the length of that line segment, depending on the context.
- A *cevian* of a triangle is a line segment from a vertex to the opposite side.
- We use the notation $\angle XYZ$ to denote either the angle between XY and YZ or the measure of that angle, depending on the context.

- The notation $[XYZ]$ denotes the area of $\triangle XYZ$.
- The notation $O(r)$ refers to the circle centered at point O with radius r . The circle may sometimes also be referred to as circle O .
- If \widehat{XY} is an arc of a circle, then $m(\widehat{XY})$ denotes the circular measure of that arc. The arc extends counterclockwise along the circle from X to Y .
- Typically, we use r for the inradius of a triangle and w for the inradius of a skewed sector.

2. INRADIUS FORMULA

Formulas for the radius of the circle inscribed in a skewed sector and in a triangle are known. Since these are not well-known, we review them here.

Theorem 2.1 (Inradius of skewed sector). *Let APB be a skewed sector and let \mathcal{C} be the circle associated with arc \widehat{AB} . Suppose P lies inside \mathcal{C} . Let R be the radius of \mathcal{C} , let w be the radius of the circle inscribed in the skewed sector, and let r be the radius of the circle inscribed in $\triangle APB$. Extend BP to meet the circle \mathcal{C} at C and draw AC . Let $\alpha, \beta, \gamma, \delta$, and ϵ be the measures of five angles associated with the skewed sector as shown in Figure 4. Then*

$$w = \frac{4R \sin \frac{\beta}{2} \sin \frac{\gamma}{2} \cos \frac{\delta}{2} \sin \frac{\epsilon}{2}}{\left(\cos \frac{\alpha}{2}\right)^2},$$

$$r = \frac{4R \sin \frac{\beta}{2} \sin \frac{\gamma}{2} \sin \frac{\epsilon}{2} \cos \frac{\epsilon}{2}}{\cos \frac{\alpha}{2}}.$$

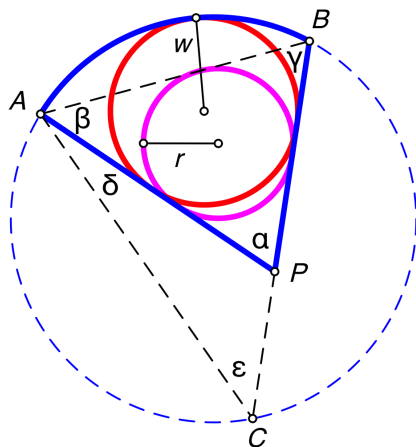


FIGURE 4. angles associated with a skewed sector

Proof. See [4, pp. 96–97] or [11, p. 26]. □

An immediate consequence of this theorem is the following result.

Theorem 2.2. *Let APB be a skewed sector and suppose P lies inside the associated circle. Let w be the radius of the circle inscribed in the skewed sector, and let r be the radius of the circle inscribed in $\triangle APB$. Let α be the vertex angle of the skewed sector, let θ_1 be the arc angle of the skewed sector, and let θ_2 be the arc angle of the opposite skewed sector. See Figure 5. Then*

$$\frac{w}{r} = \frac{\cos(\theta_2/4)}{\cos(\alpha/2) \cos(\theta_1/4)}.$$

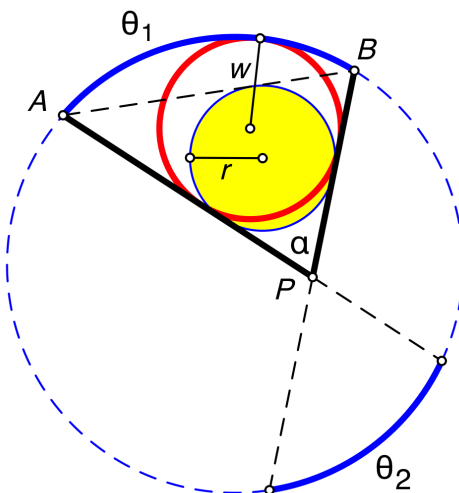


FIGURE 5.

We can also express w/r without using θ_2 as follows.

Theorem 2.3. *Using the same notation as in Theorem 2.2,*

$$\frac{w}{r} = 1 + \tan(\alpha/2) \tan(\theta_1/4).$$

Proof. See [11, pp. 26–27]. □

3. RELATIONSHIP BETWEEN INCIRCLES OF SKEWED SECTORS AND INCIRCLES OF TRIANGLES

To prove a relationship between skewed sector inradii, Theorems 2.1, 2.2, or 2.3 could be used to find the length of each radius. This is a brute force technique and better methods are available. One strategy for finding relationships between the radii of circles inscribed in skewed sectors is to relate these circles to circles inscribed in triangles, for which results are already known.

Here are some theorems that relate circles in skewed sectors to circles in triangles.

The following theorem appeared on a tablet in 1781. See [4, problem 4.0.3], [3, problem 2.2.8], [5], and [9].

Theorem 3.1 (Ajima’s Theorem). *Let AB be a chord of a circle and let C be a point inside the circle, not on the chord. See Figure 6. Let $W(w)$ be the incircle*

of skewed sector ACB (the red circle) and let $O(r)$ be the incircle of $\triangle ACB$ (the yellow circle). Then

$$w = r + \frac{2d(s-a)(s-b)}{cs},$$

where d is the height of the segment formed by AB , $a = BC$, $b = AC$, $c = AB$, and s is the semiperimeter of $\triangle ABC$.

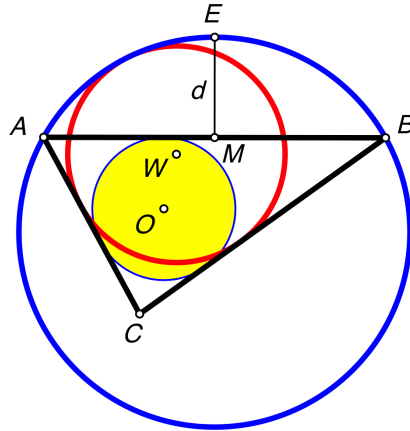


FIGURE 6. $w = r + 2d(s-a)(s-b)/(cs)$

Proof. See [4, pp. 96-97]. A more detailed proof can be found in [10, pp. 40-49]. \square

Theorem 3.2. Let D be any point on side BC of $\triangle ABC$. Cevian AD extended meets the circumcircle of $\triangle ABC$ at D' . Let $W_1(w_1)$ be the incircle of skewed sector BDD' and let $W_2(w_2)$ be the incircle of skewed sector CDD' . Let $O_1(r_1)$ be the incircle of $\triangle ADB$ and let $O_2(r_2)$ be the incircle of $\triangle ADC$ (Figure 7). Then

$$\frac{1}{r_1} + \frac{1}{w_2} = \frac{1}{r_2} + \frac{1}{w_1}.$$

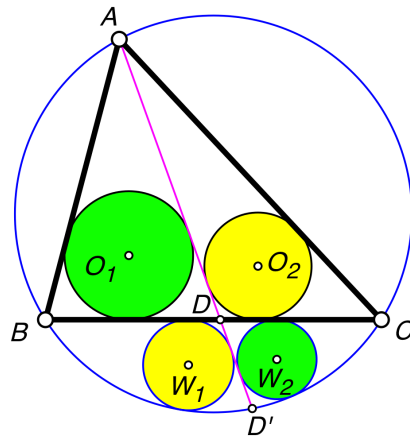


FIGURE 7. $1/r_1 + 1/w_2 = 1/r_2 + 1/w_1$

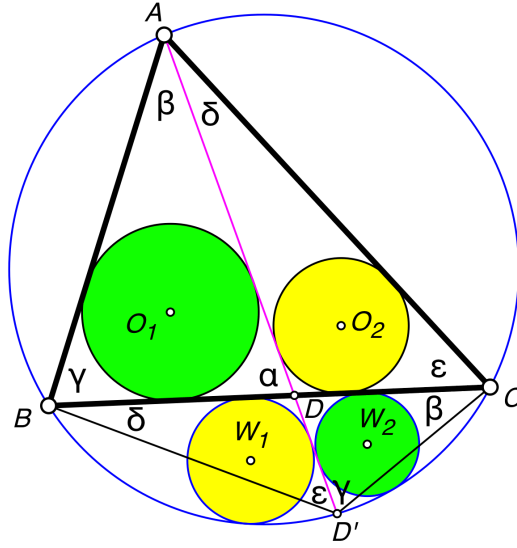


FIGURE 8. angle names

Proof. We give names to the various angles as shown in Figure 8.

Note that $\angle CDA = \alpha_2$ is supplementary to $\angle BDA = \alpha$, so $\cos(\alpha_2/2) = \sin(\alpha/2)$. Four applications of Theorem 2.1 gives

$$w_1 = \frac{4R \sin(\beta/2) \cos(\gamma/2) \sin(\delta/2) \sin(\epsilon/2)}{\sin^2(\alpha/2)},$$

$$w_2 = \frac{4R \sin(\beta/2) \sin(\gamma/2) \sin(\delta/2) \cos(\epsilon/2)}{\cos^2(\alpha/2)},$$

$$r_1 = \frac{4R \sin(\beta/2) \sin(\gamma/2) \sin(\epsilon/2) \cos(\epsilon/2)}{\cos(\alpha/2)},$$

$$r_2 = \frac{4R \sin(\delta/2) \sin(\epsilon/2) \sin(\gamma/2) \cos(\gamma/2)}{\cos(\alpha/2)}.$$

Now form the expression

$$S = \frac{1}{w_1} + \frac{1}{r_2} - \left(\frac{1}{w_2} + \frac{1}{r_1} \right).$$

Then substitute $\alpha = \delta + \epsilon$ and then $\epsilon = \pi - \beta - \gamma - \delta$. Simplifying the resulting expression (using a computer algebra system), shows that $S = 0$. \square

The result of Theorem 3.2 is so elegant that it is unlikely that it is true only because the complicated trigonometric expression, S , in the proof just happens to simplify to 0.

Open Question. *Is there a simple proof of Theorem 3.2 that does not involve a large amount of trigonometric computation requiring computer simplification?*

The following theorem appeared on a tablet in 1844 in the Aichi prefecture. See [3, problem 1.4.7] and [2, p. 22].

Theorem 3.3. *Chords AB and CD of a circle meet at E . Let $W_1(w_1)$ be the incircle of skewed sector BED and let $W_2(w_2)$ be the incircle of skewed sector AEC . Let $O_1(r_1)$ be the incircle of $\triangle BED$ and let $O_2(r_2)$ be the incircle of $\triangle AEC$. See Figure 9. Then*

$$\frac{1}{r_1} + \frac{1}{w_2} = \frac{1}{r_2} + \frac{1}{w_1}.$$

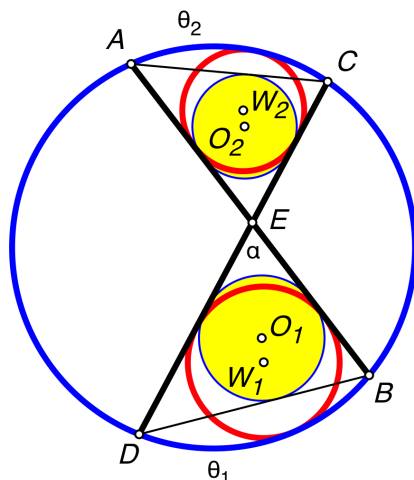


FIGURE 9. $1/r_1 + 1/w_2 = 1/r_2 + 1/w_1$

Proof. This proof comes from [11, pp. 26–27]. Let α be the vertex angle of skewed sector BED . Let θ_1 be its arc angle and let θ_2 be the arc angle of the opposite skewed sector AEC . By Theorem 2.3, we have

$$\frac{w_1}{r_1} = 1 + \tan(\alpha/2) \tan(\theta_1/4)$$

which is equivalent to

$$\frac{w_1}{r_1} - 1 = \tan(\alpha/2) \tan(\theta_1/4)$$

or

$$\frac{1}{r_1} - \frac{1}{w_1} = \frac{\tan(\alpha/2) \tan(\theta_1/4)}{w_1}.$$

Similarly,

$$\frac{1}{r_2} - \frac{1}{w_2} = \frac{\tan(\alpha/2) \tan(\theta_2/4)}{w_2}.$$

But

$$\frac{\tan(\theta_1/4)}{w_1} = \frac{\tan(\theta_2/4)}{w_2}$$

by Theorem 4.2 (which will be proved in the next section). Thus,

$$\frac{1}{r_1} - \frac{1}{w_1} = \frac{1}{r_2} - \frac{1}{w_2}$$

and the result follows. \square

Theorem 3.4. *Chords AB and CD of a circle meet at E , with $\angle AEC = \alpha$. Let $W_1(w_1)$ be the incircle of skewed sector BED and let $W_2(w_2)$ be the incircle of skewed sector AEC . Let $O_1(r_1)$ be the incircle of $\triangle BED$ and let $O_2(r_2)$ be the incircle of $\triangle AEC$. See Figure 9. Then*

$$r_1 r_2 = w_1 w_2 \cos^2 \frac{\alpha}{2}.$$

Proof. Let $m(\widehat{DB}) = \theta_1$ and $m(\widehat{CA}) = \theta_2$. Applying Theorem 2.2 to skewed sector BED gives

$$\frac{w_1}{r_1} = \frac{\cos(\theta_2/4)}{\cos(\alpha/2) \cos(\theta_1/4)}.$$

Applying Theorem 2.2 to skewed sector AEC gives

$$\frac{w_2}{r_2} = \frac{\cos(\theta_1/4)}{\cos(\alpha/2) \cos(\theta_2/4)}.$$

Multiplying these two equations gives

$$\frac{w_1 w_2}{r_1 r_2} = \frac{1}{\cos^2(\alpha/2)}$$

and the result follows by cross-multiplying. □

Theorem 3.5. *Cevians AD and CF in $\triangle ABC$ meet at P and $\angle BFC = \angle BDA$. The cevians are extended to meet the circumcircle of $\triangle ABC$ at points D' and F' , respectively, as shown in Figure 10. Let $W_1(w_1)$ be the incircle of skewed sector BDD' and let $W_2(w_2)$ be the incircle of skewed sector BFF' . Let $O_1(r_1)$ be the incircle of $\triangle BDD'$ and let $O_2(r_2)$ be the incircle of $\triangle BFF'$. Then*

$$\frac{w_1}{r_1} = \frac{w_2}{r_2}.$$

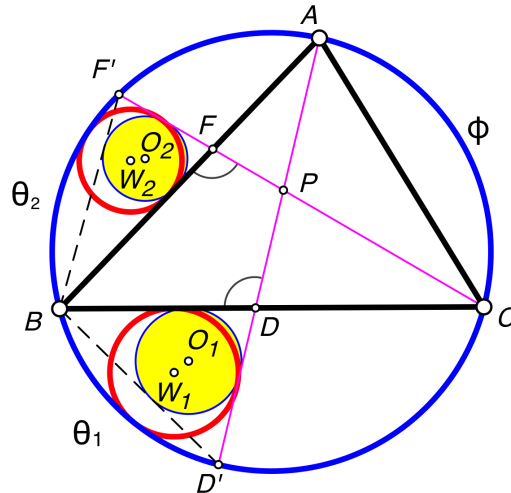


FIGURE 10. $w_1/r_1 = w_2/r_2$

Proof. Let $m(\widehat{BD'}) = \theta_1$, $m(\widehat{F'B}) = \theta_2$, and $m(\widehat{CA}) = \phi$. Applying Theorem 2.2 to skewed sector BDD' using $\alpha_1 = \angle BDD'$ gives

$$\frac{w_1}{r_1} = \frac{\cos(\phi/4)}{\cos(\alpha_1/2) \cos(\theta_1/4)}.$$

Applying Theorem 2.2 to skewed sector BFF' using $\alpha_2 = \angle BFF'$ gives

$$\frac{w_2}{r_2} = \frac{\cos(\phi/4)}{\cos(\alpha_2/2) \cos(\theta_2/4)}.$$

But $\alpha_1 = \alpha_2$ because they are supplementary to the two given angles. Chords AD' and BC intercept arcs of measures θ_1 and ϕ , so $\alpha_1 = (\theta_1 + \phi)/2$. Similarly, $\alpha_2 = (\theta_2 + \phi)/2$. Thus, $\theta_1 = \theta_2$ because $\alpha_1 = \alpha_2$. Therefore, $w_1/r_1 = w_2/r_2$. \square

Theorem 3.6. *Let H be the orthocenter of acute triangle ABC . The altitudes AD and CF are extended to meet the circumcircle of $\triangle ABC$ at points D' and F' , respectively. Let $W_1(w_1)$ be the incircle of skewed sector BDD' and let $W_2(w_2)$ be the incircle of skewed sector BFF' . Let $O_1(r_1)$ be the incircle of $\triangle BDH$ and let $O_2(r_2)$ be the incircle of $\triangle BFH$. See Figure 11. Then*

$$\frac{w_1}{r_1} = \frac{w_2}{r_2}.$$

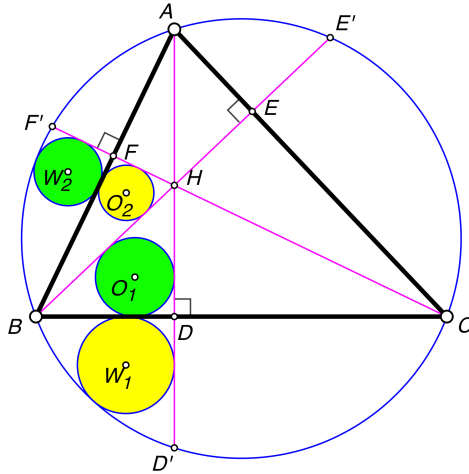


FIGURE 11. $w_1/r_1 = w_2/r_2$

Proof. Let r'_1 be the inradius of $\triangle BDD'$ and let r'_2 be the inradius of $\triangle BFF'$. Since $\angle BFC = \angle BDA$, by Theorem 3.5, we have $w_1/r'_1 = w_2/r'_2$. Now $\angle CBD' = \angle CAD'$ since both angles subtend the same arc. But $\angle CAD' = \angle CBE'$ since both angles are complementary to $\angle ACB$. Thus, $\angle DBD' = \angle DBH$. Right triangles BDD' and BDH share a common side. Thus $\triangle BDD' \cong \triangle BDH$. Hence $r_1 = r'_1$. Similarly, $r_2 = r'_2$. Therefore, $w_1/r_1 = w_2/r_2$. \square

Lemma 3.7. *Let H be the orthocenter of acute $\triangle ABC$, and let the altitudes be AD , BE , and CF as shown in Figure 12. Circles $O_1(r_1)$, $O_2(r_2)$, $O_3(r_3)$, and $O_4(r_4)$ are inscribed in triangles BHD , BHF , CAF , and ACD , respectively. Then $r_1/r_2 = r_4/r_3$.*

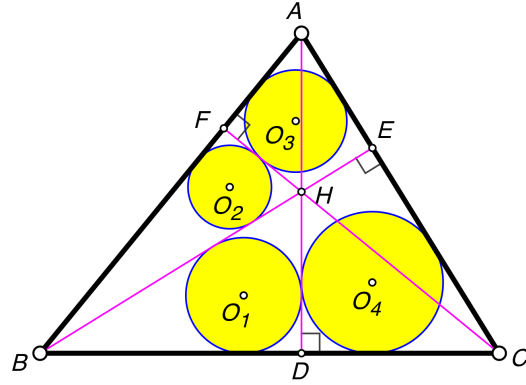


FIGURE 12. $r_1/r_2 = r_4/r_3$

Proof. Note that $\triangle BHF \sim \triangle CAF$. Therefore the figure consisting of $\triangle BHF$ and its incircle is similar to the figure consisting of $\triangle CAF$ and its incircle. Corresponding parts of similar figures are in proportion, so $r_2/r_3 = BH/CA$. In the same manner, $\triangle BHD \sim \triangle ACD$ which implies that $r_1/r_4 = BH/AC$. Therefore, $r_2/r_3 = r_1/r_4$ or $r_1/r_2 = r_4/r_3$. \square

Corollary 3.8. *Let H be the orthocenter of acute $\triangle ABC$. The altitudes AD and CF are extended to meet the circumcircle of $\triangle ABC$ at points D' and F' , respectively. Let $W_1(w_1)$ be the incircle of skewed sector BDD' and let $W_2(w_2)$ be the incircle of skewed sector BFF' . Let $O_3(r_3)$ be the incircle of $\triangle AFC$ and let $O_4(r_4)$ be the incircle of $\triangle ADC$. See Figure 13. Then $w_1/w_2 = r_4/r_3$.*

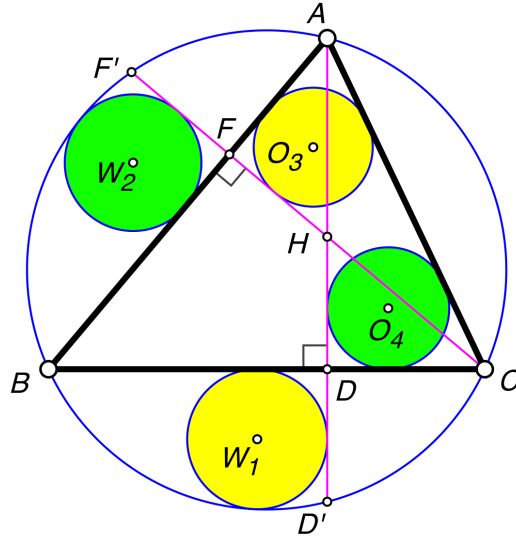


FIGURE 13. $w_1/w_2 = r_4/r_3$

Proof. By Theorem 3.6, $w_1/w_2 = r_1/r_2$. By Lemma 3.7, $r_1/r_2 = r_4/r_3$. Therefore, $w_1/w_2 = r_4/r_3$. \square

4. RELATIONSHIPS BETWEEN THE INCIRCLES OF TWO SKEWED SECTORS

Theorem 4.1. *Chords AB and CD of a circle meet at E . Let $W_1(w_1)$ be the circle inscribed in skewed sector DEB and let $W_2(w_2)$ be circle inscribed in skewed sector AEC , as shown in Figure 14. Then*

$$\frac{w_1}{w_2} = \frac{\tan(\theta_1/2)}{\tan(\theta_2/2)}$$

where $\angle BCD = \theta_1$ and $\angle ADC = \theta_2$.

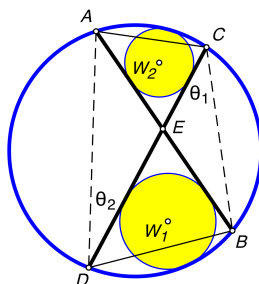


FIGURE 14. $w_1/w_2 = \tan(\theta_1/2)/\tan(\theta_2/2)$

Proof. See [4, pp. 96–97] or [11, p. 26–27]. □

Since the measure of an angle inscribed in a circle is half the circular measure of the intercepted arc, we have the following result.

Theorem 4.2. *Chords AB and CD of a circle meet at E . Let $W_1(w_1)$ be the circle inscribed in skewed sector DEB and let $W_2(w_2)$ be the circle inscribed in skewed sector AEC , as shown in Figure 15. Then*

$$\frac{w_1}{w_2} = \frac{\tan(\theta_1/4)}{\tan(\theta_2/4)}$$

where $m(\widehat{CA}) = \theta_1$ and $m(\widehat{DB}) = \theta_2$.

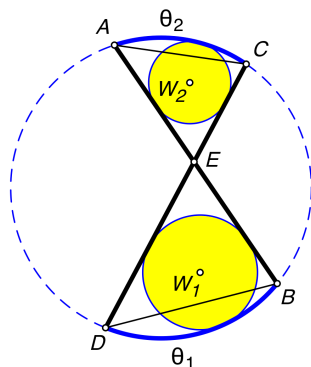


FIGURE 15. $w_1/w_2 = \tan(\theta_1/4)/\tan(\theta_2/4)$

The reader may wonder if there is any geometric significance to the angle $\theta_1/4$. If M is the midpoint of arc \widehat{DB} , then $\angle BDM = \frac{1}{2} \widehat{MB}$ and $\widehat{MB} = \frac{1}{2} \widehat{DB} = \frac{1}{2}\theta_1$, so $\angle BDM = \theta_1/4$.

There is a related result involving the incircles of triangles.

Theorem 4.3. *Chords B_1B_2 and C_1C_2 of a circle meet at A . Let r_1 and r_2 be the inradii of $\triangle B_1AC_1$ and $\triangle B_2AC_2$, respectively, as shown in Figure 16. Let $B_1C_1 = a_1$ and let $B_2C_2 = a_2$. Then $r_1/r_2 = a_1/a_2$.*

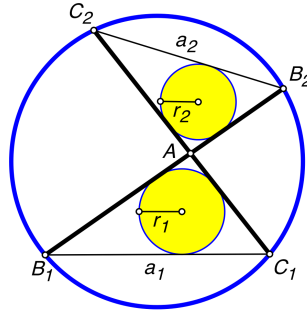


FIGURE 16. $r_1/r_2 = a_1/a_2$

Proof. This follows from the fact that $\triangle B_1AC_1 \sim \triangle B_2AC_2$. □

The following theorem comes from [11, Problem 21] and is related to Ajima's Theorem.

Theorem 4.4. *Chords B_1B_2 and C_1C_2 of a circle meet at A . Let $W_1(w_1)$ be the circle inscribed in skewed sector B_1AC_1 and let $W_2(w_2)$ be the circle inscribed in skewed sector B_2AC_2 . Let v_1 and v_2 be the heights of the segments formed by chords B_1C_1 and B_2C_2 as shown in Figure 17. Then*

$$\frac{w_1}{w_2} = \frac{v_1 a_2}{v_2 a_1}$$

where $a_1 = B_1C_1$ and $a_2 = B_2C_2$.

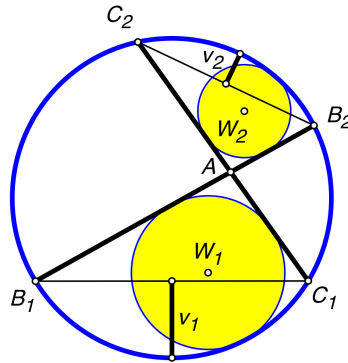


FIGURE 17. $w_1/w_2 = v_1 a_2 / v_2 a_1$

The following result is due to Pohoatza and Ehrmann, [6].

Theorem 4.5. *Let D be the point on side BC of $\triangle ABC$ such that $AB + BD = AC + CD$. A circle is circumscribed about $\triangle ABC$. Let $W_1(w_1)$ be the circle inscribed in skewed sector ADB and let $W_2(w_2)$ be the circle inscribed in skewed sector ADC (Figure 18). Then $w_1 = w_2$.*

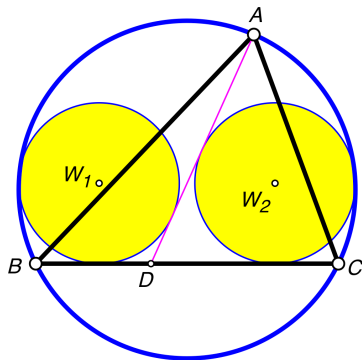


FIGURE 18. $w_1 = w_2$

Proof. Extend AD to meet the circumcircle of $\triangle ABC$ at D' . Let $O_1(r_1)$ be the circle inscribed in $\triangle BDD'$ and let $O_2(r_2)$ be the circle inscribed in $\triangle CDD'$ (Figure 19). Then $1/r_1 + 1/w_2 = 1/r_2 + 1/w_1$ by Theorem 3.2 (with points A and D' interchanged). But $r_1 = r_2$ by Theorem 3.4 of [8]. Therefore, $w_1 = w_2$. \square

See [1] for another proof.

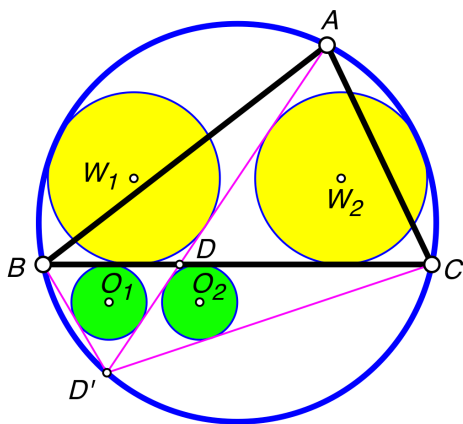


FIGURE 19. $r_1 = r_2$ and $w_1 = w_2$

5. RELATIONSHIPS BETWEEN THE INCIRCLES OF SIX SKEWED SECTORS

Theorem 5.1. *Let H be the orthocenter of acute $\triangle ABC$. The altitudes through H extended to meet the circumcircle of $\triangle ABC$ divide the interior of that circumcircle into six skewed sectors, each with vertex at H , as shown in Figure 20. Let $W_i(w_i)$ be the circle tangent to two altitudes and internally tangent to the circumcircle as shown. Then $w_1w_3w_5 = w_2w_4w_6$.*

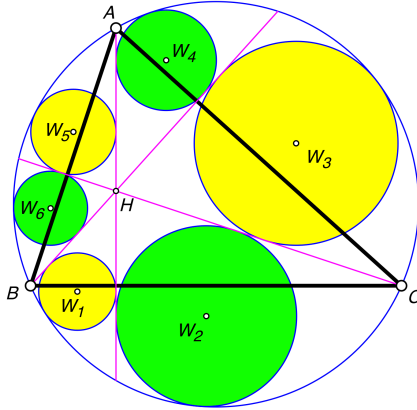


FIGURE 20. $w_1w_3w_5 = w_2w_4w_6$

Proof. Let θ_i be the arc angle of the skewed sector containing circle W_i . By Theorem 4.2, $w_1/w_4 = \tan(\theta_1/4)/\tan(\theta_4/4)$. Similarly, $w_2/w_5 = \tan(\theta_2/4)/\tan(\theta_5/4)$ and $w_3/w_6 = \tan(\theta_3/4)/\tan(\theta_6/4)$. Consequently,

$$\frac{w_1w_3w_5}{w_2w_4w_6} = \frac{w_1}{w_4} \cdot \frac{w_3}{w_6} \cdot \frac{w_5}{w_2} = \frac{\tan(\theta_1/4)}{\tan(\theta_4/4)} \cdot \frac{\tan(\theta_3/4)}{\tan(\theta_6/4)} \cdot \frac{\tan(\theta_5/4)}{\tan(\theta_2/4)}.$$

Note that $\angle BAH = \angle BCH$ since both are complementary to $\angle ABC$. Therefore, $\theta_1 = \theta_6$. Similarly, $\theta_2 = \theta_3$ and $\theta_4 = \theta_5$. Hence

$$\frac{w_1w_3w_5}{w_2w_4w_6} = \frac{\tan(\theta_1/4)}{\tan(\theta_5/4)} \cdot \frac{\tan(\theta_3/4)}{\tan(\theta_1/4)} \cdot \frac{\tan(\theta_5/4)}{\tan(\theta_3/4)} = 1,$$

so $w_1w_3w_5 = w_2w_4w_6$. □

Theorem 5.2. *Let I be the incenter of $\triangle ABC$. The cevians through I extended to meet the circumcircle of $\triangle ABC$ divide the interior of that circumcircle into six skewed sectors, each with vertex at I , as shown in Figure 21. Let $W_i(w_i)$ be the circle tangent to two cevians and internally tangent to the circumcircle as shown. Then $w_1w_3w_5 = w_2w_4w_6$.*

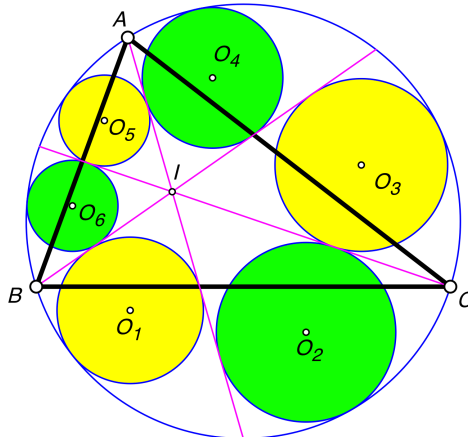


FIGURE 21. $w_1w_3w_5 = w_2w_4w_6$

Proof. Let θ_i be the arc angle of the skewed sector containing circle $W_i(w_i)$. By Theorem 4.2, $w_1/w_4 = \tan(\theta_1/4)/\tan(\theta_4/4)$. Similarly, $w_2/w_5 = \tan(\theta_2/4)/\tan(\theta_5/4)$ and $w_3/w_6 = \tan(\theta_3/4)/\tan(\theta_6/4)$. Consequently,

$$\frac{w_1 w_3 w_5}{w_2 w_4 w_6} = \frac{w_1}{w_4} \cdot \frac{w_3}{w_6} \cdot \frac{w_5}{w_2} = \frac{\tan(\theta_1/4)}{\tan(\theta_4/4)} \cdot \frac{\tan(\theta_3/4)}{\tan(\theta_6/4)} \cdot \frac{\tan(\theta_5/4)}{\tan(\theta_2/4)}.$$

Note that $\angle BAI = \angle CAI$ since AI is an angle bisector. Therefore, $\theta_1 = \theta_2$. Similarly, $\theta_3 = \theta_4$ and $\theta_5 = \theta_6$. Hence

$$\frac{w_1 w_3 w_5}{w_2 w_4 w_6} = \frac{\tan(\theta_1/4)}{\tan(\theta_3/4)} \cdot \frac{\tan(\theta_3/4)}{\tan(\theta_5/4)} \cdot \frac{\tan(\theta_5/4)}{\tan(\theta_1/4)} = 1,$$

so $w_1 w_3 w_5 = w_2 w_4 w_6$. □

Theorem 5.3. *Let H be the orthocenter of acute $\triangle ABC$. The altitudes through H extended to meet the circumcircle of $\triangle ABC$ divide the segments of the circumcircle bounded by the sides of the triangle into two skewed sectors each as shown in Figure 22. Let $W_i(w_i)$ be the incircles of the six skewed sectors formed, situated as shown in Figure 22. Then $w_1 w_3 w_5 = w_2 w_4 w_6$.*

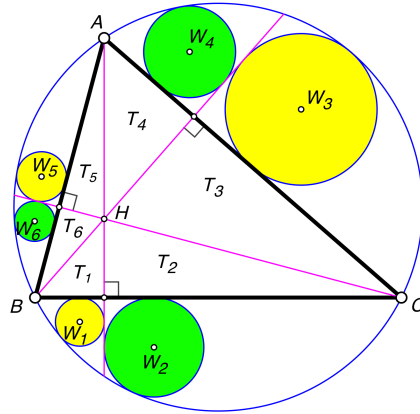


FIGURE 22. $w_1 w_3 w_5 = w_2 w_4 w_6$

Proof. The altitudes of $\triangle ABC$ divide it into six triangles named T_1 through T_6 as shown in Figure 22. Let r_i be the inradius of triangle T_i . By Theorem 3.6, $w_1/r_1 = w_6/r_6$. Similarly, $w_3/r_3 = w_2/r_2$ and $w_5/r_5 = w_4/r_4$. Therefore,

$$\frac{w_1 w_3 w_5}{r_1 r_3 r_5} = \frac{w_6 w_2 w_4}{r_6 r_2 r_4}.$$

But $r_1 r_3 r_5 = r_2 r_4 r_6$ by Theorem 3.1 of [7]. Therefore, $w_1 w_3 w_5 = w_2 w_4 w_6$. □

Theorem 5.4. *Let M be the centroid of $\triangle ABC$. The medians through M extended to meet the circumcircle of $\triangle ABC$ divide the segments of the circumcircle bounded by the sides of the triangle into two skewed sectors each as shown in Figure 23. Let $W_i(w_i)$ be the incircles of the six skewed sectors formed, situated as shown in Figure 23. Then*

$$\frac{1}{w_1} + \frac{1}{w_3} + \frac{1}{w_5} = \frac{1}{w_2} + \frac{1}{w_4} + \frac{1}{w_6}.$$

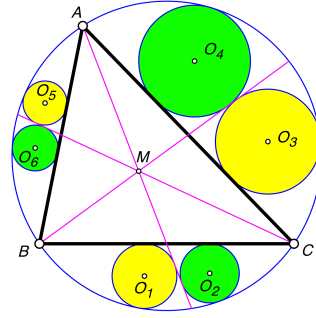


FIGURE 23. $1/w_1 + 1/w_3 + 1/w_5 = 1/w_2 + 1/w_4 + 1/w_6$

Proof. A cevian through a point P inside a triangle ABC divides $\triangle ABC$ into two triangles, known as *side triangles*. There are six such side triangles, named S_1 through S_6 as shown in Figure 24.

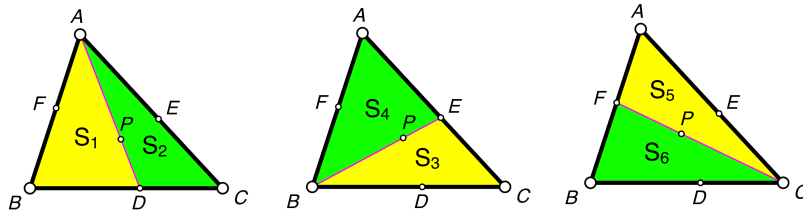


FIGURE 24. naming of side triangles

Let r_i be the radius of the circle inscribed in triangle S_i . When P is the centroid of $\triangle ABC$, Theorem 2.2 from [8] states that

$$\frac{1}{r_1} + \frac{1}{r_3} + \frac{1}{r_5} = \frac{1}{r_2} + \frac{1}{r_4} + \frac{1}{r_6}.$$

By Theorem 3.2,

$$\begin{aligned} & \left(\frac{1}{w_1} - \frac{1}{w_2} \right) + \left(\frac{1}{w_3} - \frac{1}{w_4} \right) + \left(\frac{1}{w_5} - \frac{1}{w_6} \right) \\ &= \left(\frac{1}{r_1} - \frac{1}{r_2} \right) + \left(\frac{1}{r_3} - \frac{1}{r_4} \right) + \left(\frac{1}{r_5} - \frac{1}{r_6} \right) = 0, \end{aligned}$$

so $1/w_1 + 1/w_3 + 1/w_5 = 1/w_2 + 1/w_4 + 1/w_6$. □

Theorem 5.5. *Let H be the orthocenter of acute $\triangle ABC$. The altitudes through H divide the triangle into six side triangles, S_1 through S_6 as shown in Figure 24. Let $W_i(w_i)$ be the incircle of the skewed sector associated with S_i . Two of these circles are shown in Figure 25. Then $w_1w_3w_5 = w_2w_4w_6$.*

Proof. This follows from Theorem 5.3 by applying Corollary 3.8. □

Theorem 5.6. *Let O be the circumcenter of $\triangle ABC$. The cevians through O extended to meet the circumcircle of $\triangle ABC$ divide the interior of that circumcircle into six skewed sectors, each having vertex at O , as shown in Figure 26. Let $W_i(w_i)$*

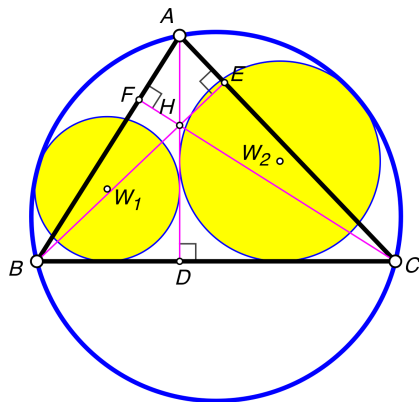


FIGURE 25. $w_1w_3w_5 = w_2w_4w_6$

be the circle tangent to two cevians and internally tangent to the circumcircle as shown. Then

$$w_1 = w_4, \quad w_2 = w_5, \quad w_3 = w_6.$$

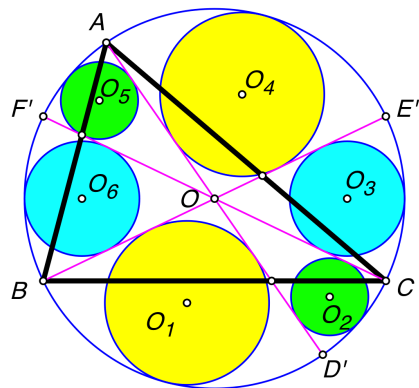
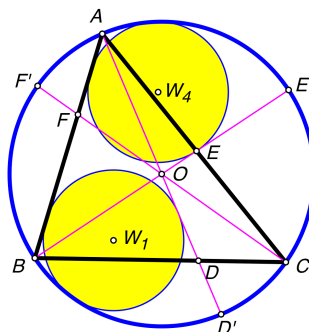


FIGURE 26. $w_1 = w_4, w_2 = w_5, w_3 = w_6$

Proof. It suffices to show that $w_1 = w_4$. Note that $\angle BOD' = \angle AOE'$ because they are vertical angles. Also, $OB = OD' = OE' = OA$ because they are all radii of circle O . Therefore, skewed sectors BOD' and AOE' are congruent and thus their incircles are also congruent. \square

Theorem 5.7. Let O be the circumcenter of $\triangle ABC$. The cevians through O are extended to meet the circumcircle of $\triangle ABC$ at points D' , E' , and F' as shown in Figure 27. The cevians divide $\triangle ABC$ into six side triangles named S_1 through S_6 as shown in Figure 24. Six circles, $W_i(w_i)$, are inscribed in the skewed sectors associated with these side triangles. Two of these circles are shown in Figure 27. Then

$$w_1 = w_4, \quad w_2 = w_5, \quad w_3 = w_6.$$

FIGURE 27. $w_1 = w_4$

Proof. It suffices to show that $w_1 = w_4$. Note that $OA = OB$ because they are both radii of circle O . Thus, $\angle OAB = \angle OBA$ because they are the base angles of an isosceles triangle. Also, $AD' = BE'$ because they are both diameters of circle O . The skewed sectors BAD' and ABE' have side AB in common. Therefore, skewed sectors BAD' and ABE' are congruent and hence their incircles are also congruent. \square

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