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The arbelos in Wasan geometry: Atsumi's problem with division by zero calculus

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Abstract. We generalize a sangaku problem involving an arbelos proposed by Atsumi by division by zero and division by zero calculus.

Keywords. arbelos, sangaku, oriented circle, oriented line, division by zero, division by zero calculus.

Mathematics Subject Classification (2010). 01A27, 51M04.

1. Introduction

For a point C on the segment AB such that $|BC| = 2a$, $|CA| = 2b$, we consider an arbelos formed by the three semicircles α, β and γ of diameters BC, CA and AB, respectively, constructed on the same side of AB (see Figure 1). The radical axis of α and β is called the axis, and the point of intersection of the semicircle γ and the axis is denoted by I.

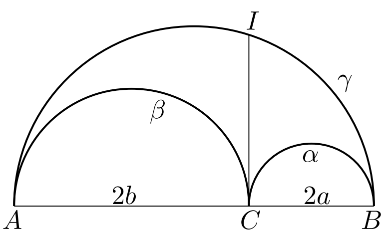

Figure 1.

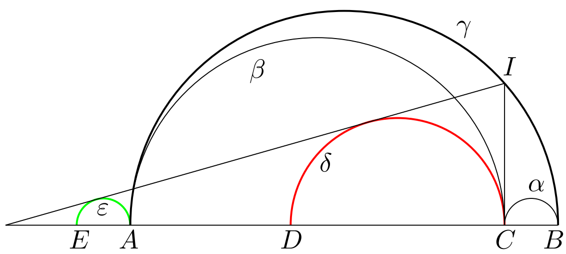

Figure 2.

Atsumi (厚海弥蔵) proposed the following problem with a numerical solution $(a, b, d, e) = (1, 7, 4, 1)$ in a sangaku hung in 1878 [1] (see Figure 2).

Problem 1. For points D and E on the line AB lying in the order E, A, D, C, let δ and ε be the circles of radii d and e and diameters CD and AE, respectively. If $a = e$ and one of the external common tangents of δ and ε passes through the point I, then show that the relation $b = d - a + d^2/(4a)$ holds.

In this paper we generalize the problem, and consider circles touching a line passing through the point I in a general way. We use a rectangular coordinate system with origin C such that the farthest point on α from AB has coordinates (a, a).

2. GENERALIZATION

We generalize Problem 1. To demonstrate our generalization we consider circles and lines with orientations, which are denoted by arrows on the figures. Two oriented figures are said to touch if they touch as usual figures and the orientations at the point of tangency are the same. We consider that an oriented circle has signed radius, whose sign is plus if the orientation is counterclockwise otherwise minus.

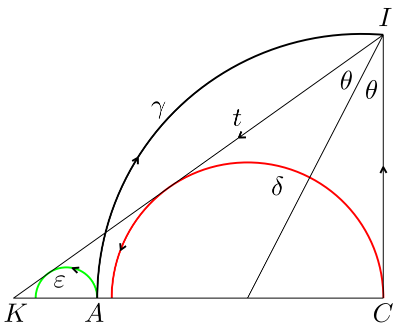

Figure 3: $d > 0$, $e > 0$, $x = |AK|$.

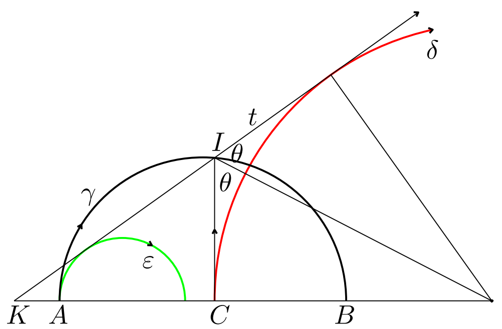

Figure 4: $d < 0$, $e < 0$, $x = |AK|$.

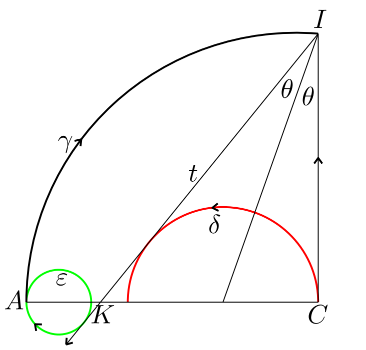

Figure 5: $d > 0$, $e < 0$, $x = -|AK|$.

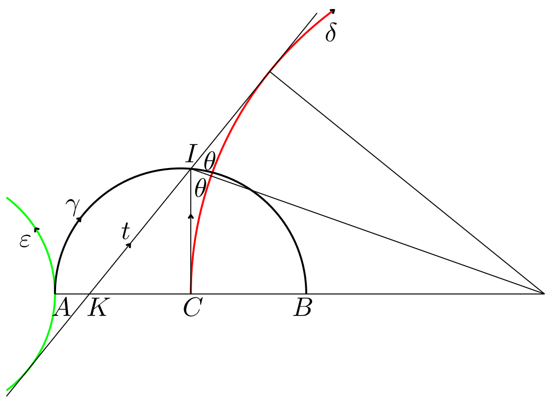

Figure 6: $d < 0$, $e > 0$, $x = -|AK|$.

Theorem 1. *Let t be an oriented line passing through the point I different from the axis. Assume that the axis has orientation from C to I and γ has clockwise orientation. If δ is the oriented circle of radius d touching t and the axis at C, and ε is the oriented circle of radius e touching t and γ at A, then we have*

$$b = d - e + \frac{d^2}{4a}. \tag{1}$$

Proof. Let θ be the angle between the axis and the line joining I and the center of δ, and $s = \tan\theta$. Considering the power of the point C with respect to γ, we have $|CI| = 2\sqrt{ab}$. Hence if t is parallel to AB, then $d = e = \pm 2\sqrt{ab}$, which satisfy (1). Assume that t meets AB in a point K. Firstly we consider the case in which the point A lies between K and C (see Figures 3 and 4) or K lies between A and C (see Figures 5 and 6). Let $x = \sigma|AK|$, where $\sigma = 1$ if $\overrightarrow{CA}$ and $\overrightarrow{AK}$ have the same orientation otherwise -1. If t has orientation from I to K (see Figures 3 and 5), then we have

$$s = \frac{d}{2\sqrt{ab}}, \quad \frac{2b+x}{2\sqrt{ab}} = \frac{2s}{1-s^2}, \quad \frac{d}{2b+x} = \frac{e}{x}.$$

If t has orientation from K to I (see Figures 4 and 6), then $d < 0$ and we have

$$s = \frac{-d}{2\sqrt{ab}}, \quad \frac{2b+x}{2\sqrt{ab}} = -\frac{2s}{1-s^2}, \quad \frac{d}{2b+x} = \frac{e}{x}.$$

Notice that the last three equations are obtained from the former by changing the signs of s. Eliminating s and x from the three equations we get (1).

Secondly we consider the case in which K lies between B and C (see Figures 7 and 8) or B lies between C and K (see Figures 9 and 10). Let $x = \sigma|BK|$, where $\sigma = 1$ if $\overrightarrow{CB}$ and $\overrightarrow{BK}$ have the same orientation, otherwise -1. If t has orientation from I to K (see Figures 7 and 9), then we have

$$s = \frac{-d}{2\sqrt{ab}}, \quad \frac{2a+x}{2\sqrt{ab}} = \frac{2s}{1-s^2}, \quad \frac{d}{2a+x} = \frac{e}{2(a+b)+x}.$$

If t has orientation from K to I (see Figures 8 and 10), then we have

$$s = \frac{d}{2\sqrt{ab}}, \quad \frac{2a+x}{2\sqrt{ab}} = \frac{-2s}{1-s^2}, \quad \frac{d}{2a+x} = \frac{e}{2(a+b)+x}.$$

Eliminating s and x from each of the three equations, we also get (1). □

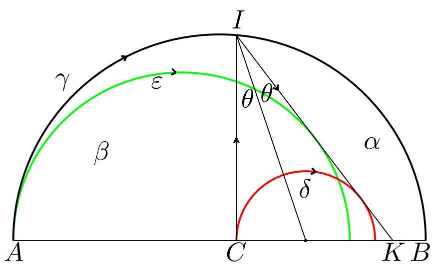

Figure 7: $d < 0$, $e < 0$, $x = -|BK|$.

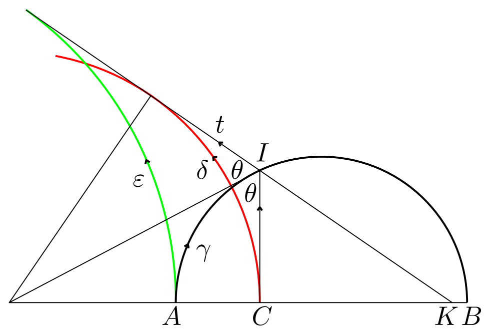

Figure 8: $d > 0$, $e > 0$, $x = -|BK|$.

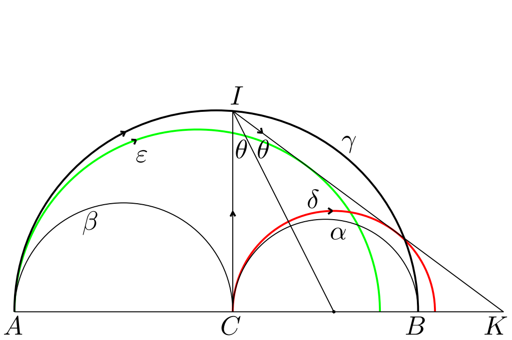

Figure 9: $d < 0$, $e < 0$, $x = |BK|$.

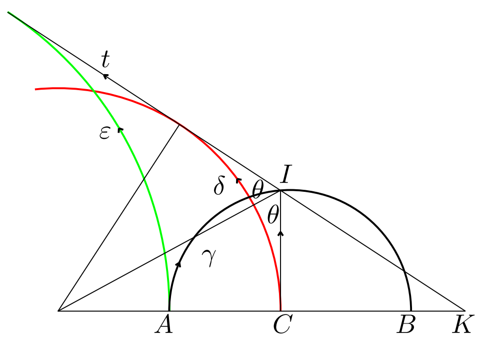

Figure 10: $d > 0$, $e > 0$, $x = |BK|$.

Corollary 1. *The following statements hold.*
(i) *If a circle of radius e touches γ externally at A and the remaining tangent of β from I, then $b^2 = 4ae$ holds.*
(ii) *If a circle touches γ internally at A and the remaining tangent of α from I,*

then it passes through the midpoint of the segment joining B and the center of α.
(iii) *The circle touching α internally at C and the tangent of γ at I has radius $2a$.*
(iv) *The circle touching α internally at C and the remaining tangent of β from I has radius $4a$.*

Proof. If β and δ overlap in Theorem 1, then $b = d$. This proves (i) by (1) (see Figure 11). If δ and α overlap, then $d = -a$, which implies $2(a+b) = -2e + a/2$ by (1), where $e < 0$. This proves (ii) (see Figure 12). If γ and ε overlap, then $e = -(a+b)$, which implies $d = -2a$ by (1). This proves (iii) (see Figure 13). If β and ε overlap, then $e = -b$, which give $d = -4a$ by (1). This proves (iv) (see Figure 14). □

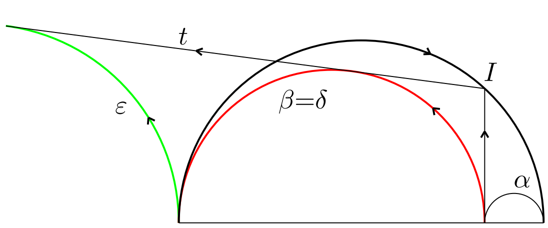

Figure 11: $b^2 = 4ae$.

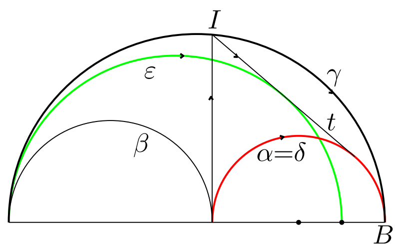

Figure 12: $-2e = 2(a+b) - \frac{a}{2}$.

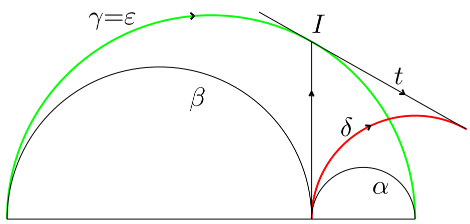

Figure 13: $d = -2a$

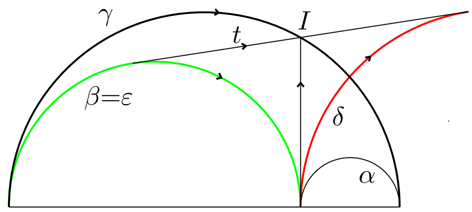

Figure 14: $d = -4a$

The part (i) is a special case of a generalized case considered in [8]. The part (iii) is stated in [19].

3. Division by zero and division by zero calculus

We did not consider the case in which the line t and the axis overlap in Theorem 1. The case will be considered in the next section using recently made definitions of division by zero and division by zero calculus [3], [20]. In this section we briefly introduce the two definitions.

For a field F there is a canonical bijection $\psi : F \to F$ such that

$$\psi(a) = \begin{cases} a^{-1} & \text{if } a \neq 0 \\ 0 & \text{if } a = 0. \end{cases}$$

It is a custom to denote $z\psi(a)$ by z/a if $a \neq 0$, i.e., $z\psi(a) = a/z$ for $a \neq 0$. Following to this, we write

$$z \cdot \psi(0) = \frac{z}{0} \quad for\ \forall z \in F.$$

Then we have

$$z \cdot \psi(a) = \frac{z}{a} \quad for\ \forall a, z \in F.$$

Especially we have

$$\frac{z}{0} = z \cdot 0 = 0 \quad for\ \forall z \in F. \tag{2}$$

Notice that the concept of the reduction to common denominator can not be used for $z/0$, i.e., we have the following relation in general in the case $b = 0$ or $d = 0$:

$$\frac{a}{b} + \frac{c}{d} \neq \frac{ad + bc}{bd}.$$

We also use the definition $f(a) = c_0$ for a function $f(x)$ having Laurent expansion

$$f(x) = \sum_{i=-\infty}^{\infty} c_k (x - a)^k$$

about $x = a$. This is a generalization of (2) called division by zero calculus.

4. The case t and the axis overlapping

We now consider the excluded case in which t and the axis overlap in Theorem 1. There are two cases: (ci) t has orientation from I to C, and (cii) t has orientation from C to I. Firstly we consider the case (ci), which is obtained if $\theta = 0$ in Figures 5 and 7. Hence δ coincides with C and ε overlaps with β (see Figure 15). Hence we get $(d, e) = (0, -b)$, which satisfies (1), i.e., Theorem 1 holds in this case.

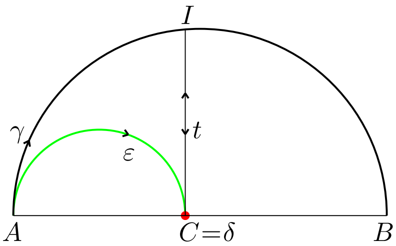

Figure 15: $(d, e) = (0, -b)$.

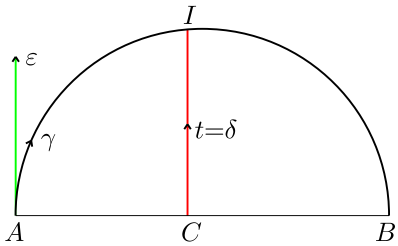

Figure 16: $(d, e) = (0, 0)$.

We consider the case (cii). Most mathematicians may consider that δ coincides with t and ε is the perpendiculars to AB at A in this case (see Figure 16). While we can consider that *a line has radius* 0 *as a circle*. For a line and a circle are represented by the equation $p(x^2 + y^2) - 2qx - 2ry + s = 0$, and if the equation represents a circle, it radius equals

$$\sqrt{\frac{q^2 + r^2 - ps}{p^2}}. \tag{3}$$

A line is obtained if $p = 0$ and (3) equals 0 in the same case by (2). Therefore we can consider that a line has radius 0 as a circle. This implies $(d, e) = (0, 0)$. However it does not satisfy (1), i.e., Theorem 1 is not true in the case (cii), if we consider by division by zero.

We consider the case (cii) by division by zero calculus. Assume that t meets the segment AB in a point K. Removing the arrows from Figures 6 and 8, we consider the case $|CK| = 0$. Let $k = |CK|$ and D and E be the centers of δ and ε, respectively. From the three similar right triangles having hypotenuses DK, EK and IK (see Figure 17)), we have

$$\frac{d}{d + k} = \frac{e}{2b - k + e} = \frac{2\sqrt{ab}}{\sqrt{4ab + k^2}},$$

where notice that $|IK| = \sqrt{4ab+k^2}$. Solving the equations for d and e, we get

$$d = \frac{2\sqrt{ab}(2\sqrt{ab}+\sqrt{4ab+k^2})}{k} \quad \textit{and} \quad e = \frac{2\sqrt{ab}(2b-k)}{\sqrt{4ab+k^2}-2\sqrt{ab}}. \tag{4}$$

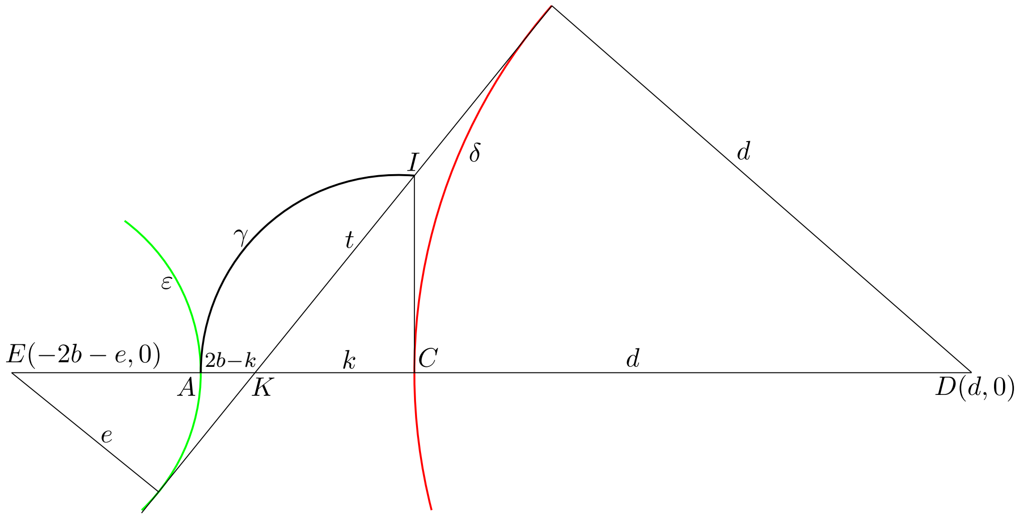

Figure 17.

Therefore using (4), the circles δ and ε are represented by the following equations in terms of a, b and k, respectively:

$$\delta(x,y;k) = (x-d)^2+y^2-d^2 \quad \textit{and} \quad \varepsilon(x,y;k) = (x-(-2b-e))^2+y^2-e^2.$$

Then the Laurent expansions of those functions about $k=0$ are

$$\delta(x,y;k) = -\frac{2^4abx}{k}+(x^2+y^2)-xk+\frac{x}{2^4ab}k^3-\frac{x}{2^7a^2b^2}k^5+\cdots,$$

$$\varepsilon(x,y;k) = \frac{2^5ab^2(x+2b)}{k^2}-\frac{2^4ab(x+2b)}{k}+((x+3b)^2+y^2-b^2)-(x+2b)k+\cdots.$$

Hence we get

$$\delta(x,y;0) = x^2+y^2 \quad \textit{and} \quad \varepsilon(x,y;0) = (x+3b)^2+y^2-b^2.$$

Therefore if $k=0$, then the circle δ coincides with the point C, and ε is the circle of radius b and center of coordinates $(-3b,0)$. Therefore ε and t do not touch even as non-oriented figures, a contradiction (see Figure 18). This implies that the hypothesis of Theorem 1 is not true in the case (cii). In this sense the Theorem 1 still holds in this case by division by zero calculus. We now see that Theorem 1 holds even if t overlaps with the axis by division by zero and division by zero calculus.

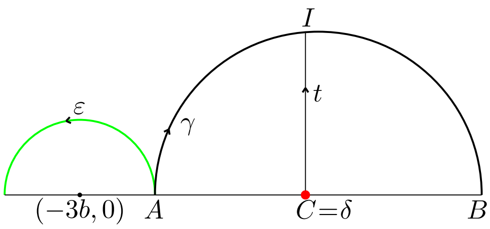

Figure 18.

The definitions of division by zero and division by zero calculus were founded by Saburou Saitoh. The two definitions enable us to consider singular cases which have never been considered up to present time. Saburou Saitoh has been making a list of successful example applying division by zero and division by zero calculus, and there are more than 1200 evidences in the list. It shows that a new world of mathematics can be opened if we introduce them. For an extensive reference of division by zero and division by zero calculus including those evidences, see [20]. For more application of division by zero and division by zero calculus to circle geometry and Wasan geometry see [2], [4], [5, 6, 7, 8, 9, 10, 11, 12, 13, 14], [15, 16, 17, 18].

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