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How *Wasanka* Did Integration: The Case of the Japanese Wedge

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Abstract. I discuss solutions of two related problems transcribed in [6, pp. 393–94, 408–12] about a surface I call a Japanese wedge, which resembles but is crucially different from the conocuneus of Wallis. These solutions illustrate how Japanese of the early 19th century numerically evaluated definite integrals using infinite series. I also show that certain tear-drop shaped sections of the Japanese wedge differ from pegtop curves except in special circumstances.

Keywords. conoid, wedge, pegtop, definite integral, infinite series, *wasan*

Mathematics Subject Classification (2020). 01A27, 51M25

1. INTRODUCTION

An elliptical CONOID is a ruled surface in which every point on an ellipse (the DIRECTOR)—possibly a circle—lies on a line (a RULING) that passes through a fixed line (the AXIS or EDGE), which is parallel to the plane of the director (the BASE). Crucially, all the rulings are parallel to a plane (a DIRECTRIX) orthogonal to the base; because of this condition, elliptical conoids are Catalan surfaces, as explained below. If the edge is perpendicular to a directrix, the surface is a RIGHT CONOID. In 1684, John Wallis called the right circular conoid a *shipwright's wedge*, or, as he put it in Latin, CONOCUNEUS [17].

Let the base of an elliptical conoid be the XY plane of a three-dimensional Cartesian coordinate system with its origin at the center of the director. If the director is an ellipse other than a circle, we stipulate that its major and minor axes lie, respectively, on the X and Y axes of the system.

It is usual to stipulate further that the edge of a right elliptical wedge is bisected by the Z axis. Hence, if the edge and major axis (or, in the case of a circle, parallel diameter) are equal, their endpoints are the vertices of a rectangle in the XZ plane.

In the shape the Japanese called a *daen-setsu* 'elliptical wedge' 楕円楔, the quadrilateral in the XZ plane is, by contrast, typically an isosceles trapezoid, with unequal parallel sides. For this reason, it is not a conoid though it resembles one and gets it name from the same sort of tool that inspired Wallis. In modern geometry, the term 'wedge' has come to be used in several unrelated ways, so to be precise, I suggest that the shape in Figure 1 be called a *Japanese wedge* to distinguish it from the wedge that Wallis studied.

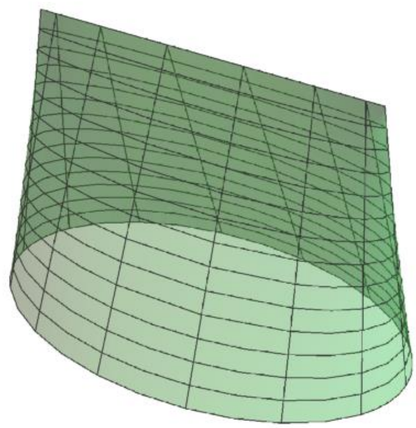

Figure 1. Japanese Wedge

As a ruled surface, a Japanese wedge may be described by the two-parameter vector equation $\boldsymbol{w}(u,v) = \boldsymbol{\alpha}(u) + v\boldsymbol{\gamma}(u)$, where $\boldsymbol{\alpha}(u) = (a\cos u\,, b\sin u\,, 0)$ and $\boldsymbol{\gamma}(u) = \big((a - c)\ \cos u\,, b\sin u\,, -h\big)$. Since $\boldsymbol{\gamma}(u)\cdot\boldsymbol{\gamma}'(u)\times\boldsymbol{\gamma}''(u) = \det\big(\boldsymbol{\gamma}(u), \boldsymbol{\gamma}'(u), \boldsymbol{\gamma}''(u)\big) = b(-a + c)h$ is zero if and only if $a = c$, $\boldsymbol{w}$ is not a Catalan surface, unlike the conocuneus.[2] Even if one gives a conocuneus an elliptical rather than circular director, the Japanese wedge is mathematically distinct.

Hasegawa Hiromu 長谷川弘 (1810‐1887), who discussed both kinds of wedges ([9], [13, pp. 284–90]), understood this difference.[3] In [9], he discusses, among many other topics, problems about both Japanese wedges and conocunei, which he calls *sakkei* 作形 'artificial; fictive' wedges. In such a wedge, every directrix plane cuts the wedge in an isosceles triangle. If, as Hasegawa usually assumes, the director is a circle and the edge equals its diameter, all the triangles have the same altitude and one has Wallis's conocuneus. But if, as in Figure 2, the

[2] It is also not a developable surface—it cannot be flattened onto a plane without distortion—because $\det\big(\boldsymbol{\alpha}'(u), \boldsymbol{\gamma}'(u), \boldsymbol{\gamma}(u)\big) = bch\cos u\sin u$ is zero if and only if $u = n\pi/2$, $n \in \mathbf{Z}$.

[3] An eight-member study group of the longstanding *Kinki wasan zemināru* (Kansai Region Traditional Mathematics Seminar) met bimonthly starting in 2001 to discuss Hasegawa's treatise, and, in or after 2005, issued a detailed 230-page annotated edition in modern Japanese compiled under the direction of Kotera Hiroshi [11].

director is an ellipse with major axis longer than the edge, then for directrices that do not intersect the edge between its endpoints, the altitudes of the isosceles triangles get shorter as one approaches the endpoints of the major axis, where they are zero.

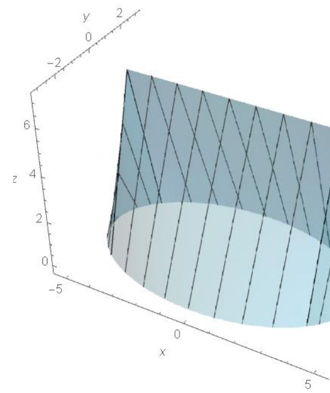

Figure 2. An "artificial" wedge

For Hasegawa, a *seikei* 正形 or 'proper; normal' wedge (our Japanese wedge) is defined differently. Assuming again a circular director with a diameter equal to the edge, one divides the edge and the two semicircular director arcs between the endpoints of the diameter parallel to the edge into n equal parts each; joins the points on the two arcs to the corresponding points on the edge with rulings; and imagines n increasing indefinitely. In modern terms, that is exactly the operation described by the equation for $\boldsymbol{w}(u, v)$ above. Except for the rulings with $y = 0$, none is parallel to a directrix, as can be seen in Figure 3.

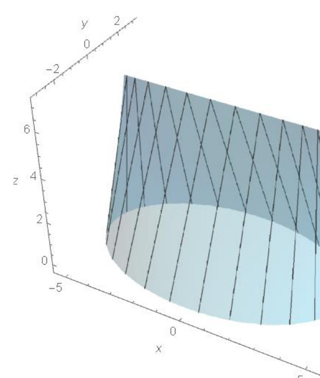

Figure 3. A "proper" wedge

Equivalent parameterizations of a Japanese wedge W for ellipse axes $2a > 2b$, edge $2c$, and altitude h are $(x, y, z) = \big((a - at + ct)\cos u\,, b(1 - t)\sin u\,, ht\big)$ for all $u, 0 \le u < 2\pi$, and $t, 0 \le t \le 1$; and, from the standard Cartesian equation for ellipses, $(x, y, z) = \left(x, \frac{\pm b(1-t)\sqrt{(a-(a-c)t)^2 - x^2}}{a-(a-c)t}, ht\right)$ for $t, 0 \le t \le 1$ and $x, -a \le x \le a$.

In fascicle 2 of [9], Hasegawa treats various plane sections of artificial wedges, turning briefly to proper wedges only in fascicle 3. He asks for the area of the intersections of the wedge and sectioning planes and for the volumes of the parts of the wedges they separate. Both kinds of problem involve integration, and fascicle 2 begins with a series of tables (discussed in [10] and [11]) that Hasegawa used to evaluate definite integrals numerically. In [11], there is an acknowledgment of a series of articles on integration in *wasan* by Fukagawa Hidetoshi ([2], [3], [4], [5]). Not mentioned, however, is [6], also by Fukagawa, in which two similar problems are presented. The first asks for the area of the section of a proper (i.e. Japanese) wedge W and a plane passing through both the point $(-c, 0, h)$ and the tangent to the director at vertex $(a, 0,0)$ (Figure 4). Fukagawa discusses a solution that he ascribes to Yoshida Tameyuki 吉田為幸 (1819–1892). It uses the parameter t but neither of parameterizations given above. Instead, Yoshida relies on the Crossed Chords theorem, as described below, to find the lengths of chords parallel to the directrix. Significantly, Yoshida's solution appeared on a *sangaku* posted at a temple in the Yotsuya district of Edo (Tōkyō) in 1829. [4]

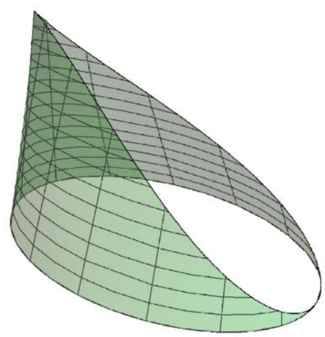

Figure 4. Special diagonal section

2. WARM-UP EXERCISE: THE SPECIAL DIAGONAL SECTION

The length of the MIDLINE (axis of symmetry) joining points $(-c, 0, h)$ and $(a, 0,0)$ is $k = \sqrt{(a+c)^2 + h^2}$.[5] A plane Π parallel to the base of W at height $z = ht, 0 \leq t \leq 1$, cuts this midline at a point P at distance kt from $(a, 0,0)$ (Figure 5). P also divides the major axis u of the ellipse in Π into segments $2a(1-t)$ and $2ct$. Likewise, the minor axis v of the ellipse in Π is $2b(1-t)$. To simplify the expression for $u = 2a(1-t) + 2ct$, Yoshida introduces $n = 1 - \frac{c}{a} = \frac{a-c}{a}$ so that he can write $u = 2a(1 - nt)$.

[4] Fukagawa refers to a commentary, which I have been unable to locate, by Yoshida (n.d.) on the book *Kokon sankan* 古今算鑑 (1832) of Uchida Yatarō 内田弥太郎 (better known as Uchida Itsumi 内田五観, 1805–1882, occasionally 内田恭). According to [1, p. 9], Yoshida posted two *sangaku* in Nagoya in 1842, but the subjects of these are not known.

[5] I generally follow Fukagawa's notation, but have adjusted it here and there to make sections 2 and 3 of this article more consistent, and to fix some slips of the pen.

Denoting the center of the ellipse in Π as O, circle $(O)\frac{u}{2}$ passes through its vertices. If the chord of this circle perpendicular to OP through P has length p, then

$$p = 2\sqrt{[(2a(1-t)][2ct]} = 4\sqrt{act(1-t)},$$

and the coincident chord of the ellipse has length

$$q = p\frac{v}{u} = \frac{2b(1-t)}{2a(1-nt)}4\sqrt{act(1-t)} = \frac{4b\sqrt{c}}{\sqrt{a}}\frac{\sqrt{t}\sqrt{1-t}^3}{1-nt}.$$

Figure 5. Ellipse and circumcircle in plane Π (2 views)

To obtain the area S of the diagonal section, we sum up the areas of all the infinitesimally narrow rectangles with long sides q lying in the diagonal plane over the length of the midline segment from the edge to the base. I.e. $S = k\left(\int_0^1 q\,dt\right) = \frac{4bk\sqrt{c}}{\sqrt{a}}\int_0^1 \frac{\sqrt{t}\sqrt{1-t}^3}{1-nt}dt$. According to [6], this leads to $S = \frac{2b\sqrt{c}k\pi}{\sqrt{a}n^3}\left(\frac{2c\sqrt{c}}{a\sqrt{a}} - 2 + 3n - \frac{3n^2}{4}\right)$ and 5.67232 for $a = 2, b = \frac{3}{2}, c = \frac{1}{2}$, and $h = 6$. A program such as Mathematica yields the more compact form $S = \frac{bk\pi(\sqrt{a}+3\sqrt{c})\sqrt{ac}}{2(\sqrt{a}+\sqrt{c})^3}$.

The crucial fact illustrated by the foregoing solution is that Hasegawa's and his predecessors knew how to evaluate definite integrals such as $\int_0^1 \frac{\sqrt{t}\sqrt{1-t}^3}{1-nt}dt$. Kotera [10] explains the series used to do this, citing [9] and modern summaries of the work of Wada Yasushi 和田寧 (or Wada Nei, 1787–1840). For instance, having found infinite series for $\pi, \pi/4, \pi^2$, and so on [13, pp. 213–17], Edo period Japanese had deduced the equivalent of $\int_0^1 \sqrt{1-x^2}\,dx = \pi/4$ [10, pp. 164–5). They evaluated $\int_0^1 \frac{1}{\sqrt{1-nt}}dt$ by using the binomial expansion of $(1-nt)^{-\frac{1}{2}}$, each term of which they could integrate [2, p. 12]. This was a remarkable achievement considering that they were sometimes less than rigorous (e.g. the series for $(1-nt)^{-\frac{1}{2}}$ does not converge unless $|nt| <1$); eschewed the use of trigonometry except for practical problems [15]; and, lacking a theory of differentiation, labored without the aid of the fundamental theorem of calculus. Nevertheless, they were able to calculate areas and volumes in many cases [8, pp.300–311], and did not shy away from difficult problems such as the ones Hasegawa systematically tackled in [9].

3. EQUATION OF A DIAGONAL SECTION

To obtain the Cartesian equation of the tear-drop-shaped section for which we just found the area, we can substitute $1-\frac{x}{k}$ for t in $y=\frac{2bk\sqrt{c}}{\sqrt{a}}\frac{\sqrt{t}\sqrt{1-t}^3}{1-nt}=\frac{4b(1-t)\sqrt{ac(1-t)t}}{a(1-t)+ct}$ provided that $0\le x\le k$.[6] Squaring, $y^2=\frac{4ab^2c(k-x)x^3}{k^2(c(k-x)+ax)^2}$. Fukagawa [7, pp. 240–42], citing a reprint of [14, v. 1, pp. 289–95], calls this a *piriforme quartique*, but that is a bit misleading for two reasons.

First, the PIRIFORM curve (less confusingly called a PEGTOP)[7] is, strictly speaking, the curve defined by the Cartesian equation $a^4y^2=b^2x^3(2a-x)$, where the pegtop's midline lies on the X axis and $y=(0,\pm b)$ for $x=a$. The equivalent parametric equations are equations $x=a(1+\sin t)$ and $y=b\cos t(1+\sin t)$ for $0\le t\le 2\pi$. Significantly, the area of a pegtop is πab, the same as that of an ellipse with semiaxes a and b; we will make use of this fact presently. For now, note that $y=\sqrt{\frac{4ab^2c(k-x)x^3}{k^2(c(k-x)+ax)^2}}=\frac{2bx\sqrt{ac(k-x)x}}{k(c(k-x)+ax)}$ yields $y=\frac{b\sqrt{ac}}{a+c}$ for $k=1$ and $x=\frac{1}{2}$.

If the curve discussed in section 2 were a pegtop, then the ellipse of equal area $S=\frac{bk\pi(\sqrt{a}+3\sqrt{c})\sqrt{ac}}{2(\sqrt{a}+\sqrt{c})^3}$ and major axis k would have semiminor axis $y=\frac{2S}{\pi k}=\frac{b(\sqrt{a}+3\sqrt{c})\sqrt{ac}}{(\sqrt{a}+\sqrt{c})^3}$. But if $k=1$, then $x=\frac{1}{2}$ so, as just noted, $\frac{b(\sqrt{a}+3\sqrt{c})\sqrt{ac}}{(\sqrt{a}+\sqrt{c})^3}=\frac{b\sqrt{ac}}{a+c}$. This is equivalent to $a=c$ and true for the conocuneus but not for the typical Japanese wedge. Curves like the one studied in section 2 are therefore pegtops only if $a=c$. Wada seems to have been the first Japanese to study curves of this kind ([16], [6, pp. 240–41]) and he called them *seitō-en* 'flame curves' 盛灯円,[8] so that is the name I will use below.

Second, the visual resemblance of flame curves and pegtops vanishes when we generalize the foregoing problem by using the point $(d,0,h)$, $-c<d<c$, rather than $(c,0,h)$ to define the diagonal plane. Taking the second parametrization in section 2 for the points on W and substituting $\frac{a-x}{a-d}$ for t, we get Wada's flame curve for $x, d\le x\le a$, but if we let x run from $-a$

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[6] We could substitute x/k for t, but since $1-t$ appears more often than t in the expression for y, our choice leads to a simpler form.

[7] Wallis, in 1685, called his quartic curve "piriform," i.e. pear-shaped, but algebraically distinct curves have more recently been given the names PEAR CURVE and PEAR-SHAPED CURVE.

[8] Sometimes the word is just *tō-en*. Hasegawa called these curves *hōshu-en* 宝珠円 [9] and Kuwamoto Masaaki called them *sen-en* 尖円 in 1855 ([13, p. 285], [11, p. 60]). *Hōshu* is the Sino-Japanese translation of Sanskrit *cintāmaṇi*, 'wish-fulfilling stone,' a flame-shaped object found in Buddhist iconography; SJ *sen* is glossed *togaru* 'sharpen (to a point)' in Japanese; SJ *en* 'circle' in these words connotes 'closed curve'.

to a, we obtain a self-intersecting closed curve[9] (Figure 6), which resembles what in astronomy is called an ANALEMMA.[10]

To digress slightly, an analemma is a map of the positions of the sun in the sky as observed from a fixed point at the same (mean solar) time over the course of a year. E.g., Figure 7 shows the points on the earth the sun is directly overhead (at 12:00 local time) during a year; the fixed point over which the sun stands at noon twice during year is approximately 10° N 85° W. For a full discussion of the spherical trigonometry and theory underlying the concept of the analemma and the calculation of its shape using astronomical data, see [12]. Though the complete flame curve resembles an analemma, it is far from clear how far one can reduce the equations of a true analemma to that of a complete flame curve.

The Cartesian equation of the flame curve is found be solving $t = \frac{a-x}{a-d}$ for x, substituting the solution, $a - at + dt$, for x in $y = \frac{\pm b(t-1)\sqrt{(a-(a-c)t)^2 - x^2}}{a-(a-c)t}$; and finally substituting $1 - \frac{x}{k}$ for t, which effectively redefines the XY plane so that the flame curve lies in that plane with its midline on the X axis and its node at the origin. Its length is $k = \sqrt{h^2 + (a-d)^2}$, and its equation is $y^2 = \frac{b^2(c-d)(k-x)x^2(c(k-x)+d(k-x)+2ax)}{k^2(c(k-x)+ax)^2}$, which reduces to the simpler $y^2 = \frac{4ab^2c(k-x)x^3}{k^2(c(k-x)+ax)^2}$ derived earlier when one substitutes $-c$ for d (the special case).

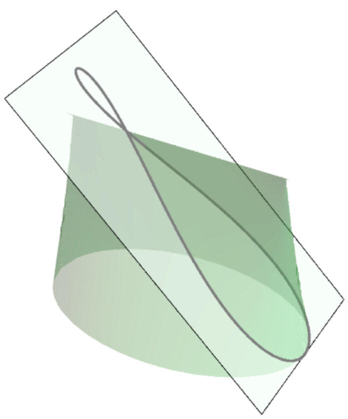

Figure 6. A complete flame curve

Figure 7. An analemma

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[9] I hesitate to call this a figure-eight curve because EIGHT CURVE and FIGURE EIGHT CURVE are both names for a different quartic, the lemniscate of Gerono, $x^4 = a^2(x^2 - y^2)$, or, in parametric form, $x = a \sin t$ and $y = a \sin t \cos t$.

[10] Mikami [13, p. 285] remarks that if the diagonal plane crosses the far end of the wedge below the edge, the curve is an egg-shaped oval without “a point or cusp,” whereas if it cuts the edge between its endpoints, the point of the curve on the edge “is not a cusp . . . but a node.” This implies that he knew the complete curve included the loop above the edge.

It will be recalled that a pegtop can be represented by the parametric equations $x = a(1 + \sin t)$ and $y = b\cos\theta(1 + \sin t)$.[11] By contrast, the equations for the flame curve for the same a and b are $x = a\left(\frac{c}{a} + \sin t\right)$ and $y = b\cos t\left(\frac{c}{a} + \sin t\right)$, with $c, -a \le c \le a$. Eliminating t, these equations lead to $a^4y^2 = b^2x^2(a^2 - (x - c)^2)$. Once again, if and only if $a = c$ do we obtain $a^4y^2 = b^2x^2(2ax - x^2)$, i.e. the Cartesian equation for the pegtop.[12]

4. MORE DIFFICULT: A DIAGONAL SECTION CUTTING THE EDGE

The problem described in [6, pp. 408–12] (Figure 8) concerns the generalization just introduced but seems to antedate the special case of $d = -c$, having been posted at a shrine in Ishikawa province in 1816. It asks for both the surface area of the flame curve and the volume of the part of the wedge cut off from the base. Fukagawa reports that, for $a = 4.5, b = 1, c = 5, h = 11$, and $c - d = 1$, Yoshida Tameyuki calculated that $S = 3.739466289$ ... and $V = 1.06577509040$ ... .

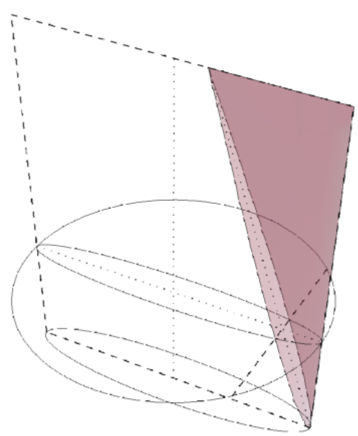

Figure 8. General diagonal section

If we proceed as before, the area of the flame curve is $S = 2k\int_0^1 \frac{y(t)\sqrt{[2x(t)-(c-d)t](c-d)t}}{x(t)}\,dt$ with $k = \sqrt{h^2 + (a-d)^2}$ and $x(t) = a - t(a - c)$, $y(t) = b(1 - t)$ for the major and minor semiaxes, respectively, of each ellipse parallel to the base of the wedge. Since we know that the

[11] N.B. The parameter t is not the angle from the X axis to the line rotated about the point $(-a, 0)$ in the locus construction of the pegtop found in standard reference works. For instance, $t = 0$ corresponds to the slope b/a, so the angle for $x = a$ is $\tan^{-1}\frac{b}{a}$.

[12] The area of a complete flame curve is $\int_0^{2\pi} b\cos t\left(\frac{c}{a} + \sin t\right)\frac{d}{dt}a\left(\frac{c}{a} + \sin t\right)dt = \int_0^{2\pi} ab\,(\cos^2 t)\left(\frac{c}{a} + \sin t\right)dt = \pi bc$, clearly not the same as the πab of the pegtop unless $a = c$. Setting the limits of integration to $(0, \pi)$ and $(\pi, 2\pi)$ gives us the respective area of each lobe of the curve, viz. $\frac{b}{6}(3\pi c + 4a)$ and $\frac{b}{6}(3\pi c - 4a)$.

area of each elliptical segment is $A = \frac{x(t)y(t)}{2}(\theta - \sin\theta)$, where θ is the central angle subtending the chord in which the diagonal and horizontal planes intersect, the volume of the wedge above the diagonal plane is $V = \int_0^1 A\,dt$.[13] Using these integrals and a program such as Mathematica, it is easy to verify that Yoshida's S and V are correct for his given values. But because of the new variable d and the need to express θ in terms of a, b, c, d, the corresponding indefinite integrals are complicated and unwieldy, certainly beyond the ability of the *wasanka* to have tackled directly. Using the standard technique of his time, Yoshida therefore converted the integrands into infinite series, each term of which he could integrate easily. In fact, he makes double use of this technique, producing for both S and Van infinite series in which each term contains an infinite series of its own.

Yoshida makes two changes in definitions to facilitate his calculations. Instead of $2c$ and $c - d$, he uses c for the whole axis and d for the part of it cut off by the diagonal plane. He also interchanges t and $1 - t$, in effect reversing the sense of the Z axis. Thus, a plane through P parallel to the base of the wedge defines an ellipse with major axis of length $c - (c - 2a)t$ and minor axis of length $2bt$.

Since P lies on the axis of the flame curve, it divides the major axis of the ellipse into segments $f = (c - 2a)t$ and $g = d(1 - t)$. Letting $n = \frac{c-2a}{c}$ and, for convenience, $w = 1 - nt$ and $x = 1 - t$, we have $g = dx, f + g = cw$, and so $f = cw - dx$. Hence the chord through P perpendicular to the major axis has length $l = \frac{2btk}{cw} \cdot 2\sqrt{(cw - dx)dx} = \frac{4btk}{cw}\sqrt{c\left(w - \frac{d}{c}x\right)dx}$. Letting $e = \frac{d}{c}$, we have $l = \frac{4btk\sqrt{dx}}{cw}\sqrt{c(w - ex)} = \frac{4btk\sqrt{dx}}{\sqrt{cw}}\sqrt{1 - \frac{e}{w}x} = \frac{4btk\sqrt{ex}}{\sqrt{w}}\sqrt{1 - \frac{e}{w}x}$. The variables w and x will soon be eliminated; Yoshida aims to produce series in $e, n,$ and t only.

He begins by writing $\sqrt{1 - \frac{e}{w}x}$ as $1 - \frac{ex}{2w} - \frac{e^2x^2}{8w^2} - \frac{e^3x^3}{16w^3} - \frac{5e^4x^4}{128w^4} - \frac{7e^5x^5}{256w^5} - \cdots$, a series he likely derived by writing out the first few terms of the binomial expansion of $\left(1 - \frac{e}{w}x\right)^p$ and then substituting $\frac{1}{2}$ for p. He next introduces $m = 8k\sqrt{e}$ so he can rewrite $\frac{4btk\sqrt{ex}}{\sqrt{w}}$ as $2btm\frac{\sqrt{x}}{4\sqrt{w}}$. Distributing $\frac{\sqrt{x}}{4\sqrt{w}}$ over the series, $l = 2btm\left(\frac{1}{4}\frac{x^{1/2}}{w^{1/2}} - \frac{ex^{3/2}}{8w^{3/2}} - \frac{e^2x^{5/2}}{32w^{5/2}} - \frac{e^3x^{7/2}}{64w^{7/2}} - \frac{5e^4x^{9/2}}{512w^{9/2}} - \frac{7e^5x^{11/2}}{1024w^{11/2}} - \cdots\right)$, or, restoring $1 - t$ and $1 - nt$ for x and w, respectively,

$$l = 2bm\left[\frac{(1-t)^{1/2}t}{4(1-nt)^{1/2}} - \frac{e(1-t)^{3/2}t}{8(1-nt)^{3/2}} - \frac{e^2(1-t)^{5/2}t}{32(1-nt)^{5/2}} - \frac{e^3(1-t)^{7/2}t}{64(1-nt)^{7/2}} - \frac{5e^4(1-t)^{9/2}t}{512(1-nt)^{9/2}} - \frac{7e^5(1-t)^{11/2}t}{1024(1-nt)^{11/2}} - \cdots\right].$$

[13] Incidentally, by setting θ to 2π, one easily finds that the volume of the whole wedge is $\frac{b\pi}{6}(2a + c)$.

Yoshida next replaces the power of $\frac{1}{\sqrt{1-nt}}$ in each term with its own infinite series:

$$\begin{aligned}l = 2bm\Bigg[&\frac{1}{4}(1-t)^{1/2}t\left(1+\frac{nt}{2}+\frac{3n^2t^2}{8}+\frac{5n^3t^3}{16}+\frac{35n^4t^4}{128}+\frac{63n^5t^5}{256}+\cdots\right)\\ &-\frac{e}{8}(1-t)^{3/2}t\left(1+\frac{3nt}{2}+\frac{15n^2t^2}{8}+\frac{35n^3t^3}{16}+\frac{315n^4t^4}{128}+\frac{693n^5t^5}{256}+\cdots\right)\\ &-\frac{e^2}{32}(1-t)^{5/2}t\left(1+\frac{5nt}{2}+\frac{35n^2t^2}{8}+\frac{105n^3t^3}{16}+\frac{1155n^4t^4}{128}+\frac{3003n^5t^5}{256}+\cdots\right)\\ &-\frac{e^3}{64}(1-t)^{7/2}t\left(1+\frac{7nt}{2}+\frac{63n^2t^2}{8}+\frac{231n^3t^3}{16}+\frac{3003n^4t^4}{128}+\frac{9009n^5t^5}{256}+\cdots\right)-\cdots\Bigg].\end{aligned}$$

Having distributed all the factors except powers of e over each series, he can integrate term by term to get

$$\begin{aligned}S = 2bm\Bigg[&\frac{1}{15}+\frac{2n}{105}+\frac{n^2}{105}+\frac{4n^3}{693}+\frac{5n^4}{1287}+\frac{2n^5}{715}\ldots\\ &-e\left(\frac{1}{70}+\frac{n}{105}+\frac{n^2}{154}+\frac{2n^3}{429}+\frac{n^4}{286}+\frac{3n^5}{1105}\ldots\right)\\ &-e^2\left(\frac{1}{504}+\frac{5n}{2772}+\frac{5n^2}{3432}+\frac{n^3}{858}+\frac{5n^4}{5304}+\frac{n^5}{1292}\ldots\right)\\ &-e^3\left(\frac{1}{1584}+\frac{7n}{10296}+\frac{7n^2}{11440}+\frac{7n^3}{13260}+\frac{7n^4}{15504}+\frac{n^5}{2584}\ldots\right)-\cdots\Bigg].\end{aligned}$$

Turning to the volume cut off by the diagonal plane, Yoshida's strategy is to slice it with planes parallel to the base of the wedge and sum up the areas of the elliptical segments formed by them. If the major axis of one of these ellipses is D and the area of the related circular segment is S', then $S'' = S'\frac{2bt}{D}$. Yoshida therefore begins with a formula for S' assuming it has a sagitta g. The formula he gives, however, seems to come out of the blue: letting $\alpha = g/D$,

$$S' = 4g\sqrt{gD}\left(\frac{1}{3}-\frac{\alpha}{10}-\frac{\alpha^2}{56}-\frac{\alpha^3}{144}-\frac{5\alpha^4}{1408}-\frac{7\alpha^5}{3328}\ldots\right).$$

Unfortunately, there is a gap at this point in [6], so we must figure out on our own how Yoshida got this series for S'. Below, I use trigonometry to do so, though Yoshida probably would have taken a different route.

Denote the series in α as s. We know the area of the segment is $\frac{D^2}{8}(\theta - \sin\theta)$ where θ is the angle subtending the segment chord. Now, $\sin\frac{\theta}{2} = \frac{\sqrt{(D-g)g}}{D/2}$ and $\cos\frac{\theta}{2} = \frac{D/2-g}{D/2}$, so, from the identity $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$, we have $\sin\theta = \frac{4(D-2g)\sqrt{(D-g)g}}{D^2}$. Substituting $D\alpha$ for g, this becomes $\sin\theta = 4(1-2\alpha)\sqrt{\alpha(1-1\alpha)}$. Denoting this as ψ, we have $\theta - \sin\theta = \sin^{-1}\psi - \psi$. Since for any real x, $\sin^{-1} x = \sum_{i=0}^{\infty} \frac{(2n-1)!!x^{2n+1}}{(2n)!!(2n+1)}$, we have $\sin^{-1}\psi - \psi = \sum_{i=1}^{\infty} \frac{(2n-1)!!\psi^{2n+1}}{(2n)!!(2n+1)}$, so $S' = \frac{D^2}{8}\left(\sum_{i=1}^{\infty} \frac{(2n-1)!!\psi^{2n+1}}{(2n)!!(2n+1)}\right) = \frac{D^2}{8}\left(\frac{32\alpha^{3/2}}{3} - \frac{16\alpha^{5/2}}{5} - \frac{4\alpha^{7/2}}{7} - \frac{2\alpha^{9/2}}{9} - \frac{5\alpha^{11/2}}{44} - \frac{7\alpha^{13/2}}{104} - \cdots\right)$. Hence $\frac{S'}{4D^2\alpha^{3/2}} = \frac{1}{3} - \frac{\alpha}{10} - \frac{\alpha^2}{56} - \frac{\alpha^3}{144} - \frac{5\alpha^4}{1408} - \frac{7\alpha^5}{3328} - \cdots = s$. Substituting g for $D\alpha$, we find that $S' = 4g\sqrt{gD}s$, just as Yoshida asserted.

Now, using the same notation as in the computation of the area S of the flame curve, Yoshida replaces g and D with dx and cw, respectively, to get

$$\begin{aligned}
S'' &= 8btdx\sqrt{cdwx}\left(\frac{1}{3} - \frac{dx}{10cw} - \frac{d^2x^2}{56c^2w^2} - \frac{d^3x^3}{144c^3w^3} - \frac{5d^4x^4}{1408c^4w^4} - \cdots\right) \\
&= 8btd\sqrt{e}\left(\frac{x^{3/2}}{3\sqrt{w}} - \frac{ex^{5/2}}{10w^{3/2}} - \frac{e^2x^{7/2}}{56w^{5/2}} - \frac{e^3x^{9/2}}{144w^{7/2}} - \frac{5e^4x^{11/2}}{1408w^{9/2}} - \cdots\right) \\
&= 32bd\sqrt{e}\left[\frac{(1-t)^{3/2}t}{12\sqrt{1-nt}} - \frac{e(1-t)^{5/2}t}{40(1-nt)^{3/2}} - \frac{e^2(1-t)^{7/2}t}{224(1-nt)^{5/2}} - \frac{e^3(1-t)^{9/2}t}{576(1-nt)^{7/2}}\right. \\
&\qquad \left. - \frac{5e^4(1-t)^{11/2}t}{5632(1-nt)^{9/2}} - \cdots\right].
\end{aligned}$$

Factoring out powers of $1-t$ and e, he expands each of $\frac{t}{12\sqrt{1-nt}}, \frac{-t}{40(1-nt)^{3/2}}, \frac{-t}{224(1-nt)^{5/2}}, \cdots$ into its own series. Letting $\omega = 8d\sqrt{e}h$, this produces

$$\begin{aligned}
S'' = 2b\omega\Bigg[&(1-t)^{3/2}\left(\frac{t}{12} + \frac{nt^2}{24} + \frac{n^2t^3}{32} + \frac{5n^3t^4}{192} + \cdots\right) \\
&-e(1-t)^{5/2}\left(\frac{t}{40} + \frac{3nt^2}{80} + \frac{3n^2t^3}{64} + \frac{7n^3t^4}{128} + \cdots\right) \\
&-e^2(1-t)^{7/2}\left(\frac{t}{224} + \frac{5nt^2}{448} + \frac{5n^2t^3}{256} + \frac{15n^3t^4}{512} + \cdots\right) \\
&-e^3(1-t)^{9/2}\left(\frac{t}{576} + \frac{7nt^2}{1152} + \frac{7n^2t^3}{512} + \frac{77n^3t^4}{3072} + \cdots\right) - \cdots\Bigg].
\end{aligned}$$

Notice that, after distributing powers of $\sqrt{1-t}$ in the series, all the integrands are of the form $n^{p-1}t^p\left(\sqrt{1-t}\right)^q$ with positive integers p, q. Yoshida is able to integrate them term by term as before and thus obtains

$$V = 2b\omega\left[\frac{1}{105} + \frac{2n}{945} + \frac{n^2}{1155} + \frac{4n^3}{9009} + \cdots\right.$$

$$-e\left(\frac{1}{630}+\frac{n}{1155}+\frac{n^2}{2002}+\frac{2n^3}{6435}+\cdots\right)$$
$$-e^2\left(\frac{1}{5544}+\frac{5n}{36036}+\frac{n^2}{10296}+\frac{n^3}{14586}+\cdots\right)$$
$$-e^3\left(\frac{1}{20592}+\frac{7n}{154440}+\frac{7n^2}{194480}+\frac{7n^3}{251940}+\cdots\right)-\cdots\Bigg].$$

As a check, I used Mathematica to verify that Yoshida's series of series indeed produce the values for S and V for his given values.

5. FINISHING TOUCH

The foregoing two problems give us a fine example of how *wasanka* went about what we retrospectively call integration and came up with tables like Hasegawa's in [9]. The solutions show that Yoshida was an intrepid algebraist—but he's not done! He concludes by taking advantage of the fact that the same chords, the lengths of which are integrated to find S, bound the elliptical segments he integrated to find V. On this basis, he describes the relationships among the coefficients that appear in the final series for S and V, respectively, by means of the following three sequences:

$$A_{-1}=0$$
$$A_0=\frac{B_0}{7} \qquad B_0=\frac{1}{15}$$
$$A_1=\frac{B_1}{9} \qquad B_1=2nA_0-q_1 \qquad q_1=\frac{e}{2\cdot 5\cdot 7}$$
$$A_2=\frac{B_2}{11} \qquad B_2=3\left(2A_1-\frac{1}{9}nA_0\right)n-q_2 \qquad q_2=\frac{1\cdot 5}{4\cdot 9}eq_1$$
$$A_3=\frac{B_3}{13} \qquad B_3=5\left(2A_2-\frac{3}{11}nA_1\right)n-q_3 \qquad q_3=\frac{3\cdot 7}{6\cdot 11}eq_2$$
$$A_4=\frac{B_4}{15} \qquad B_4=7\left(2A_3-\frac{5}{13}nA_2\right)n-q_4 \qquad q_4=\frac{5\cdot 9}{8\cdot 13}eq_3$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$A_k=\frac{B_k}{2k+7} \qquad B_k=(2k-1)\left(2A_{k-1}-\frac{2k-3}{2k+5}nA_{k-2}\right)n-q_k \qquad q_k=\frac{(2k-3)(2k+1)}{2k(2k+5)}eq_{k-1}$$

In terms of these sequences, $V=2b\omega\sum_{i=0}^{\infty}A_i$ and $S=2bm\sum_{i=0}^{\infty}B_i$. This splendid result shows how Yoshida made the most of the techniques of integration available to him with, so to speak, one hand tied behind his back: he never measures an angle, invokes a trigonometric function, or worries about convergence, yet he gets the correct numerical answers to his problems.

REFERENCES

[1] Fukagawa, Hidetoshi. 1975. Aichi no sangaku [*Sangaku* in Aichi prefecture]. *Sūgaku-shi kenkyū*, 65, 6–10.

[2] Fukagawa, Hidetoshi. 1978a. Wasan ni okeru sekibun (sono 1) [Integration in *wasan*, part 1]. *Sūgaku-shi kenkyū*, 77: 9–19.

[3] Fukagawa, Hidetoshi. 1978b. Wasan ni okeru sekibun (sono 2) [Integration in *wasan*, part 2]. *Sūgaku-shi kenkyū*, 79: 28–38.

[4] Fukagawa, Hidetoshi. 1981. Wasan ni okeru sekibun (sono 3) [Integration in *wasan*, part 3]. *Sūgaku-shi kenkyū*, 88: 9–28.

[5] Fukagawa, Hidetoshi. 1982. Wasan ni okeru sekibun (sono 4) [Integration in *wasan*, part 4]. *Sūgaku-shi kenkyū*, 95: 1–19.

[6] Fukagawa, Hidetoshi, ed. 1983. *Zoku-zoku sangaku no kenkyū* [Yet more studies of *sangaku*]. Nagoya: Narumi dofūkai.

[7] Fukagawa, Hidetoshi. 1987. Algebraic curves in Japan during the Edo period. *Historia Mathematica* 14, 235–42.

[8] Fukagawa, Hidetoshi, Rothman, Tony, 2008. *Sacred Mathematics: Japanese Temple Geometry.* Princeton University Press, Princeton.

[9] Hasegawa, Hiromu. 1844. *Sanpō kyūseki tsūkō* 求積通考 [Treatise on quadrature]. 5 fascicles. Accessible at http://www.wasan.jp/archive/kyusekituko/kyusekituko.html.

[10] Kotera, Hiroshi. 2003. Wasan ni okeru sekibun no gainen ni tsuite [On the concept of integration in *wasan*]. *Sūri kaiseki kenkyūjo kōkyūroku* [Bulletin of the Research Institute for Mathematical Sciences, Kyōto University] 1317, 162–66.

[11] Kotera, Hiroshi, ed. n.d. (2005 or later). Sanpō kyūseki tsūkō *chūshaku to kaisetsu* [Notes and commentary on *Sanpō kyūseki tsūkō*] . Kinki wasan zemināru hōkokushū 14. Accessible at same location as Hasegawa 1844.

[12] Lucht, Phil. 2013. *Sun-Earth Kinematics, the Equation of Time, Insolation and the Solar Analemma*. Available at http://user.xmission.com/~rimrock/Documents/Sun-Earth Kinematics, the Equation of Time, Insolation and the Solar Analemma.pdf.

[13] Mikami, Yoshio. 1910 [1913]. *The Development of Mathematics in China and Japan.* Reprint of the 1913 edition = *Abhandlugen zur Geschichte der mathematischen Wissenschaften*, v. 30 (Leipzig: B. G. Teubner). New York: Chelsea Publishing Co.

[14] Teixeira, F. Gomes. 1908. *Traité des courbes spéciales remarquables planes et gauches*. (Traduit de l'Espagnol, revu, and très augmenté.) Coïmbre [Portugal]: Imprimérie de l'Université.

[15] Unger, J. Marshall. 2020. On the acceptance of trigonometry in *wasan*: evidence from a text of Aida Yasuaki. *Historia Mathematica*, vol. 52 pp. 51–65.

[16] Wada, Yasushi. 1825 [1882]. *Sōsei ien sanpō* [Original methods for variant circles] 創製異円算法.

[17] Wallis, John. [1662] 1684. *Cono-cuneus: or, the Shipwright's Circular Wedge. That is, a Body resembling in part a* Conus, *in part a* Cuneus, *Geometrically considered.* London.