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Ellipses inscribed in trapezoids and solutions to 2020-1-Problems 1 and 2

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Abstract. In this article, we derive formulas regarding the relationships between lengths of sides of trapezoids and axes of their inscribed ellipses. We also explore special cases, including ellipses inscribed in isosceles trapezoids and right-angled trapezoids, respectively. Solutions to 2020-1 Problem 1 and 2 are given.

Keywords. trapezoid, ellipse.

Mathematics Subject Classification (2010). 01A27, 51M04, 51M25.

1. Introduction

In this article we consider ellipses inscribed in a trapezoid. In section 2, we derive formulas regarding the relationships between the lengths of sides of arbitrary trapezoids and the axes of their inscribed ellipses. In sections 3 and 4, we explore special cases involving ellipses inscribed in isosceles trapezoids and right-angled trapezoids, respectively. In section 5, we give solutions to Problems 1 and 2 in [1].

Problem 1. An isosceles trapezoid $WXYZ$ has bases WX and YZ ($YZ > WX$). The lateral sides XY and ZW are equal. Let $WX = s$ and $YZ = t$.
(1) A circle with radius r is inscribed in the trapezoid. Find r in terms of s and t.
(2) If an ellipse with semi-minor axis b is inscribed in the trapezoid, and the major axis of the ellipse is perpendicular to the bases of the trapezoid, find b in terms of s and t.
(3) If an ellipse with semi-major axis a is inscribed in the trapezoid, and the minor axis of the ellipse is perpendicular to the bases of the trapezoid, find a in terms of s and t.

Problem 2. An right-angled trapezoid $WXYZ$ has bases WX and YZ ($YZ > WX$). $\angle Z$ is a right angle. Let $WX = s$ and $YZ = t$.
(1) A circle with radius r is inscribed in the trapezoid. Find r in terms of s and t.
(2) If an ellipse with semi-minor axis b is inscribed in the trapezoid, and the major axis of the ellipse is perpendicular to the bases of the trapezoid, find b in terms of s and t.
(3) If an ellipse with semi-major axis a is inscribed in the trapezoid, and the minor axis of the ellipse is perpendicular to the bases of the trapezoid, find a in terms of s and t.

2. ELLIPSES INSCRIBED IN ARBITRARY TRAPEZOIDS

In this section, we consider an ellipse inscribed in trapezoid $WXYZ$ with bases WX and YZ ($YZ > WX$). We denote the lengths of the semi-axis of the ellipse that is perpendicular to the bases of the trapezoid and parallel to the bases of the trapezoid as l_1 and l_2, respectively. Let the lengths of WX, XY, YZ, ZW be s, v, t, u, respectively. Let $w = t - s$ and $z = \frac{u+v+w}{2}$ (see Figure 1).

Theorem 1. *If an axis of the ellipse is perpendicular to the bases of the trapezoid, then*

$$l_1 = \frac{\sqrt{z(z-u)(z-v)(z-w)}}{w}. \tag{1}$$

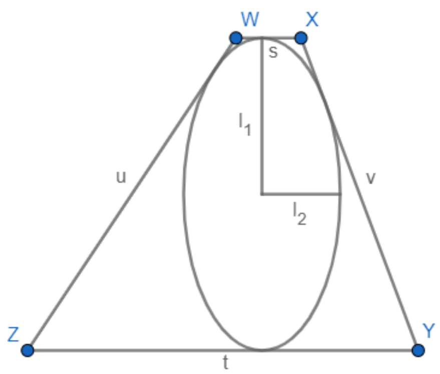

FIGURE 1. Theorems 1 and 2

We first prove the following lemma.

Lemma 1. *Let h be the height of the trapezoid, then*

$$h = \frac{2\sqrt{z(z-u)(z-v)(z-w)}}{w}. \tag{2}$$

Proof. Let P be a point on YZ such that WP is parallel to XY. $ZP = t - s = w$. Consider area of of $\triangle WPZ$, by Heron's formula,

$$\frac{hw}{2} = \sqrt{z(z-u)(z-v)(z-w)}.$$

Rearranging gives (2). ☐

Proof of Theorem 1. If an axis of the ellipse is perpendicular to the bases of the trapezoid, then the semi-axis is half of that of the height of the trapezium, $l_1 = \frac{h}{2}$. Substituting (2) in the equation gives (1). $\square$

Let $\alpha = \frac{u^2-v^2+w^2}{2w}$, $\beta = \frac{v^2-u^2+w^2}{2w}$, $\gamma = s+t$, we have the following theorem (see Figure 1).

Theorem 2. *Let $\delta = \frac{\alpha+\beta+\gamma}{2}$. If an axis of the ellipse is perpendicular to the bases of the trapezoid, then*

$$l_2 = \frac{\sqrt{-\delta(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}}{\gamma}. \tag{3}$$

To prove the theorem, we need the following lemma.

Lemma 2. *Let M and N be points on YZ such that WM and XN are perpendicular to YZ. Let $ZM = j$, $YN = k$. Then*

$$j = \alpha = \frac{u^2-v^2+w^2}{2w} \quad \text{and} \quad k = \beta = \frac{v^2-u^2+w^2}{2w}.$$

Proof. Let P be a point on YZ such that WP is parallel to XY. $ZP = t-s = w$. Consider$\triangle WPZ$, by law of cosine, $\cos\angle WZP = \frac{u^2-v^2+w^2}{2uw}$. $j = u\cos\angle WZP = \frac{u^2-v^2+w^2}{2w}$. By similar method, we can prove that $k = \frac{v^2-u^2+w^2}{2w}$. $\square$

Proof of Theorem 2. Apply an affine transformation which dilates or contracts the figure along the axis of the ellipse which is perpendicular to the bases of the trapezoid with a scale of $\frac{l_2}{l_1}$. The ellipse is transformed to a circle with radius l_2. Suppose points W, X, Y, Z, M, N are transformed to W', X', Y', Z', M', N', respectively. Note that after the transformation, $Z'M' = j = \alpha$. The height of the transformed trapezoid is $2l_2$. $X'Y' = s$ and $Y'Z' = t$. By Pythagoras' theorem, $Z'W' = \sqrt{\alpha^2+4l_2^2}$. Similarly, $X'Y' = \sqrt{\beta^2+4l_2^2}$. We know that $W'X' + Y'Z' = X'Y' + Z'W$. This is the result of Pitot's theorem, which states that in any tangential quadrilateral (i.e. one in which a circle can be inscribed), the two sums of lengths of opposite sides are equal. So $\sqrt{\alpha^2+4l_2^2} + \sqrt{\beta^2+4l_2^2} = s+t = \gamma$. Solving the equation gives (3). $\square$

3. Ellipses inscribed in isosceles trapezoids

In this section, we explore the special case where an ellipse is inscribed an isosceles trapezoid. Suppose an ellipse is inscribed in an isosceles trapezoid $WXYZ$ with bases WX and YZ ($YZ > WX$). The lateral sides XY and ZW are equal. We denote the lengths of the semi-axis of the ellipse that is perpendicular to the bases of the trapezoid and parallel to the bases of the trapezoid as l_1 and l_2, respectively. Let $WX = s$, $YZ = t$, $XY = ZW = u$ (see Figure 2).

Theorem 3. *If an axis of the ellipse is perpendicular to the bases of the trapezoid, then*

$$l_1 = \frac{\sqrt{(2u+t-s)(2u-t+s)}}{4} \tag{4}$$

and

$$l_2 = \frac{\sqrt{st}}{2}. \tag{5}$$

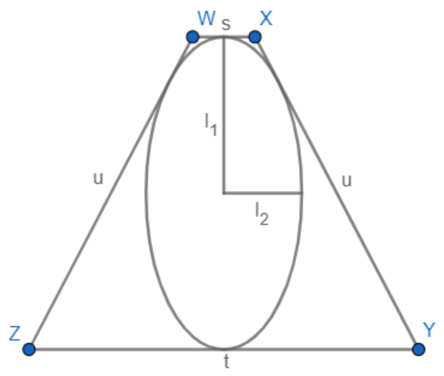

FIGURE 2. Theorem 3

We first revisit a known formula for the inradius of a tangential trapezoid [2].

Lemma 3. *If the bases of the trapezoid have lengths l and m ($l > m$), and any one of the other two sides has length n, then the inradius r is given by the formula*

$$r = \frac{\sqrt{lm(l-n)(n-m)}}{l-m}. \tag{6}$$

Proof. By Pitot's theorem, the length of the other lateral side of the trapezoid is $l + m - n$. Let ω be the diameter of the incircle which is perpendicular to the bases of the trapezoid. Consider the incircle as an inscribed ellipse with ω as its major axis. By Theorem 1, put $a = r$, $s = m$, $t = l$, $u = n$, $v = l + m - n$ into (1) gives (6). □

Proof of Theorem 3. By Theorem 1, substitute $v = u$ in (1) gives (4). Apply an affine transformation which dilates or contracts the figure along the axis of the ellipse which is perpendicular to the bases of the trapezoid with a scale of $\frac{l_2}{l_1}$. The ellipse is transformed to a circle with radius l_2. Suppose points W, X, Y, Z are transformed to W', X', Y', Z', respectively. Note that after the transformation, $W'X' = s$, $Y'Z' = t$, $X'Y' = Z'W'$. Moreover, $W'X'Y'Z'$ is a tangential trapezoid, by Pitot's theorem, $W'X' + Y'Z' = X'Y' + Z'W'$. Hence, $X'Y' = Z'W' = \frac{s+t}{2}$. By Lemma 3, substitute $r = l_2$, $l = t$, $m = s$, $n = \frac{s+t}{2}$ into (6) gives (5). □

4. ELLIPSES INSCRIBED IN RIGHT-ANGLED TRAPEZOIDS

In this section, we explore the special case where an ellipse is inscribed an right-angled trapezoid. Suppose an ellipse is inscribed in an isosceles trapezoid $WXYZ$ with bases WX and YZ ($YZ > WX$). $\angle Z$ is a right angle. The lengths of the semi-axis of the ellipse that is perpendicular to the bases of the trapezoid and parallel to the bases of the trapezoid are l_1 and l_2, respectively; and the lengths of WX, XY, YZ, ZW are s, v, t, u, respectively (see Figure 3).

Theorem 4. *If an axis of the ellipse is perpendicular to the bases of the trapezoid, then*

$$l_1 = \frac{u}{2}$$

and

$$l_2 = \frac{st}{s+t}. \tag{7}$$

Proof. It is obvious that the length of axis that is perpendicular to the bases of the trapezoid is equal to height of the trapezoid, so $l_1 = \frac{u}{2}$. Apply an affine transformation which dilates or contracts the figure along the axis of the ellipse which is perpendicular to the bases of the trapezoid with a scale of $\frac{l_2}{l_1}$. The ellipse is transformed to a circle with radius l_2. Suppose points W, X, Y, Z are transformed to W', X', Y', Z', respectively. After the transformation, $W'X' = s$, $Y'Z' = t$. Let $X'Y' = v'$ and $Z'W' = u'$. By Pythagoras theorem, $u'^2 + (t-s)^2 = v'^2$. Moreover, $W'X'Y'Z'$ is a tangential trapezoid, by Pitot's theorem, $W'X' + Y'Z' = X'Y' + Z'W'$, therefore $s + t = u' + v'$. Solving, we have $u' = \frac{2st}{s+t}$. By Lemma 3, substitute $r = l_2$, $l = t$, $m = s$, $n = \frac{2st}{s+t}$ into (6) gives (7). $\square$

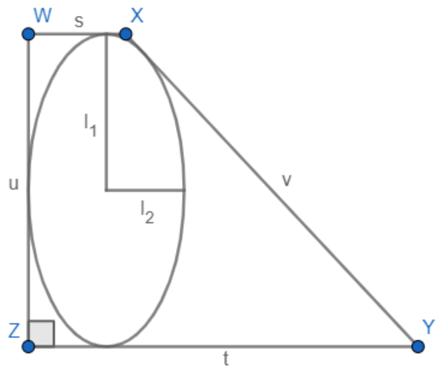

FIGURE 3. Theorem 4

5. SOLUTIONS TO 2020-1 PROBLEMS 1 AND 2

Solution of Problem 1. By Lemma 3, substitute $l = t$, $m = s$, $n = \frac{s+t}{2}$ into (6) gives $r = \frac{\sqrt{st}}{2}$. By Theorem 3, $b = l_2 = \frac{\sqrt{st}}{2}$. By Theorem 3, $a = l_2 = \frac{\sqrt{st}}{2}$. $\square$

Solution of Problem 2. By Lemma 3, substitute $l = t$, $m = s$, $n = \frac{2st}{s+t}$ into (6) gives $r = \frac{st}{s+t}$. By Theorem 4, $b = l_2 = \frac{st}{s+t}$. By Theorem 4, $a = l_2 = \frac{st}{s+t}$. $\square$

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