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A note on a generalization of a five circle problem: Part 3

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Abstract. We generalize a problem in Wasan geometry involving three smaller congruent circles touching two larger congruent circles and their common tangent.

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1. Introduction

In [3], we generalized a problem involving five circles proposed in [4]. In this note we give another generalization of the same problem. The problem is stated as follows (see Figure 1).

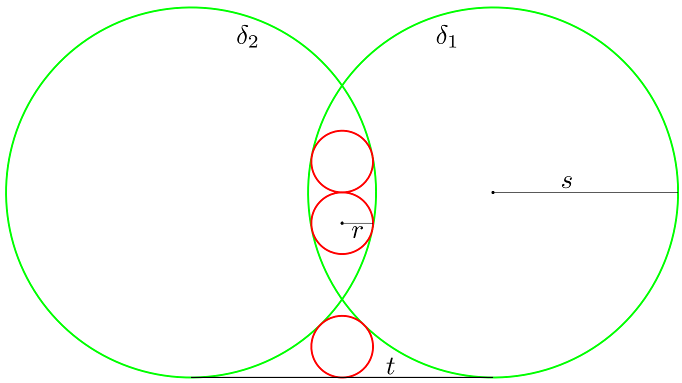

Figure 1: $s = 6r$.

Problem 1. For two intersecting circles δ_1 and δ_2 of radius s with an external common tangent t, two touching congruent circles of radius r touch δ_1 and δ_2 internally. If the inradius of the curvilinear triangle made by δ_1, δ_2 and t is also r, show that $s = 6r$.

Some generalizations of similar problems can be found in [1, 2].

2. GENERALIZATION

We generalize the problem. If γ_1, γ_2, $\cdots$, γ_n are congruent circles such that γ_1 and γ_2 touch, and γ_i touches γ_{i-1} at the farthest point on γ_{i-1} from γ_1 for $i = 3$, 4, $\cdots$, n, then the circles are said to be *congruent circles in line.*

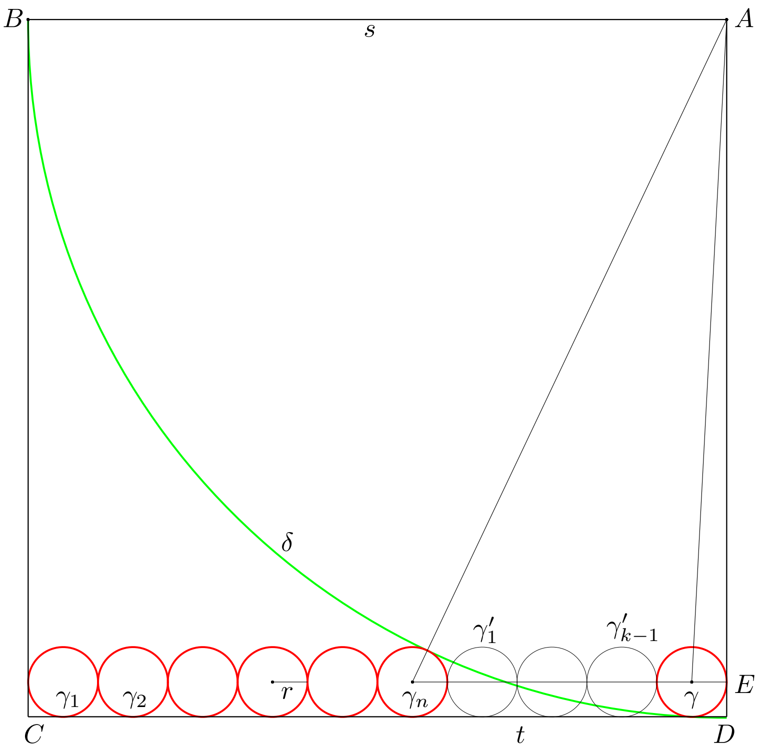

Figure 2: $n = 6$, $k = 4$.

Theorem 1. *For a rectangle $ABCD$ with $|BC| < AB| = s$, let δ be the circle of radius s and center A. Let γ_1, γ_2, $\cdots$, γ_n be congruent circles in line of radius r such that γ_1 touches the segments BC and CD, γ_n touches the segment CD from the same side as γ_1 and δ externally. If there is a circle γ of radius r touching the segments CD and DA and δ internally, the following statements hold.*
(i) $s = (2n + 1 + \sqrt{8n + 1})r$.
(ii) s/r *is an integer if and only if there is a positive integer* k *such that*

$$n = \frac{k(k-1)}{2}.$$

In this event we have

$$s = k(k+1)r,$$

and there are circles γ'_1, γ'_2, $\cdots$, γ'_{k-1} such that the circles γ_1, γ_2, $\cdots$, γ_n, γ'_1, γ'_2, $\cdots$, γ'_{k-1}, γ form congruent circles in line.

Proof. Let E be the point of tangency of γ and DA (see Figure 2). From the two right triangles, one of which is formed by A, E and the center of γ_n and the other is formed by A, E and the center of γ, we have

$$(r + s)^2 - (s - (2n - 1)r)^2 = (s - r)^2 - r^2.$$

Solving the equation for s, we have $s = (2n+1\pm\sqrt{8n + 1})r$. Since $s > 2(n+1)r > (2n + 1 - \sqrt{8n + 1})r$, (i) is proved. s/r is an integer if and only if there is a positive integer k such that $8n+1=(2k-1)^2$. The last equation is equivalent to $n=k(k-1)/2$. Substituting this in (i), we have $s=k(k+1)r$. The last part of (ii) follows from $s-2nr-2r=2(k-1)r$. □

The case of Problem 1 can be obtained if $n=1$, $k=2$.

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[4] Problem 2018–3, Sangaku J. Math., **2** (2018) 41–42.