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A configuration involving a square and two equilateral triangles

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Abstract. In this article, we investigate the geometric properties of a configuration which appeared in a sangaku problem from Yamagata hung in 1881 proposed by Gotoh[1].

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1. Introduction

In this article, we investigate the geometric properties of the following configuration (see Figure 1). Assume that $ABCD$ is a square, E and F are the points on the sides CD and DA, respectively, such that $\triangle BEF$ is an equilateral triangle, G is the midpoint of BE, H is the point on intersection of the sides EF and DG and I is the point of intersection of the sides AG and BF.

We solve the following problems proposed by Okumura [2], and also present two related results.

Problem 1.1. *Prove or disprove that the circumcircle of $\triangle EHG$ touches the lines BC, CD and AG at G.*

Problem 1.2. *Prove or disprove that the circumcircle of $\triangle FIH$ is the reflection of the circumcircle of $\triangle GHI$ in the line HI.*

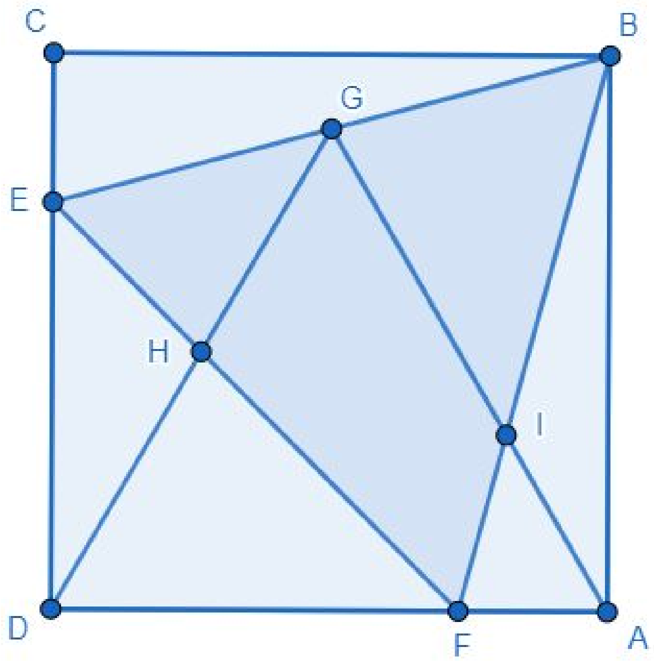

FIGURE 1. The geometric configuration which appeared in a sangaku problem proposed by Gotoh.

2. SOLUTION TO PROBLEM 1.1

We first prove an important theorem.

Theorem 2.1. *$\triangle AGD$ is an equilateral triangle.*

Proof. Since FG is perpendicular to BE, $\angle FGB + \angle BAF = 90° + 90° = 180°$. So $GFAB$ is concyclic. Then $\angle BAG = \angle BFG = \frac{60°}{2} = 30°$. Hence $\angle GAD = \angle BAD - \angle BAG = 90° - 30° = 60°$. Since G is the mid-point of BE, $\triangle AGD$ is an isosceles triangle with base DA. Therefore $\angle GDA = \angle GAD$. Therefore $\triangle AGD$ is equilateral. □

We are now ready to solve problem 1.1 (see Figure 2).

Theorem 2.2. *The circumcircle of $\triangle EHG$ has the following properties:*
(1) *It touches the line AG at G.*
(2) *It touches the line CD at E.*
(3) *It touches the line BC.*

Proof. We first prove item (1). Since $\triangle AGD$ and $\triangle BEF$ are equilateral triangles, we have $\angle HEG = \angle HGA = 60°$. Hence AG is a tangent of the circumcircle of $\triangle EHG$ at G.

We then prove item (2). $AB = AG$. $\angle AGB = \angle ABG = \frac{180° - \angle BAG}{2} = \frac{180° - 30°}{2} = 75°$. $\angle EGH = 180° - \angle AGD - \angle AGB = 180° - 60° - 75° = 45°$. Since $GFDE$ is concyclic, $\angle EFD = \angle EGD = 45°$. $\angle HED = \angle DEF = 180° - \angle EDF - \angle EFD = 180° - 90° - 45° = 45°$. So $\angle EGH = \angle HED = 45°$. CD is the tangent of circumcircle of $\triangle EHG$ at E.

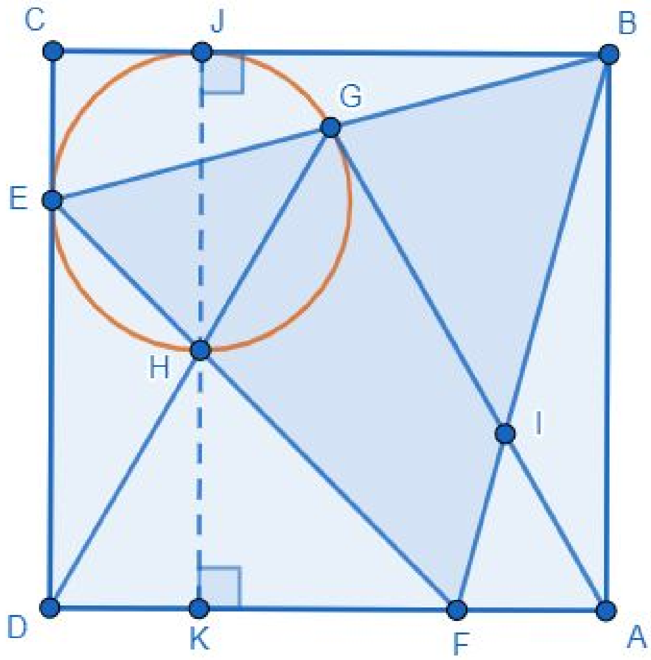

Figure 2. Theorem 2.2.

Lastly, we prove item (3). Assume that the line parallel to CD passing through H meets BC and DA in points J and K, respectively. Let $BE = l$. $\angle EHJ = \angle FHK = 180° - \angle HKF - \angle EFD = 180° - 90° - 45° = 45°$. $\angle EHG = 180° - \angle HEG - \angle EGH = 180° - 60° - 45° = 75°$. By sine law, $\frac{EG}{\sin\angle EHG} = \frac{EH}{\sin\angle EGH}$, $\frac{\frac{l}{2}}{\sin 75°} = \frac{EH}{\sin 45°}$, $EH = \frac{l\sin 45°}{2\sin 75°}$. $CJ = EH\sin\angle EHJ = \frac{l\sin 45°}{2\sin 75°}\sin 45° = \frac{\sqrt{3}-1}{2\sqrt{2}}l$. Also, $CE = BE\sin 15° = \frac{\sqrt{3}-1}{2\sqrt{2}}l$. So we have $CE = CJ$. Then $\angle CJE = 45°$. $\angle EJH = \angle CJH - \angle CJE = 90° - 45° = 45°$. Since $\angle EJH = \angle EGH = 45°$, we know that $EJGH$ are concyclic. Moreover, $\angle EHJ = \angle EJC = 45°$, so the circumcircle of $EJGH$ touches BC at J. □

This solves Problem 1.1. By Theorem 2.2, we also have the following corollary.

Corollary 2.1. *Suppose that CD and AG meet in a point M and AM meets BC in a point N, then the circumcircle of $\triangle EHG$ is the M-excircle of $\triangle CMN$.*

3. Solution to Problem 1.2 and two related results

We first prove an interesting lemma.

Lemma 3.1. *Suppose ω_1 and ω_2 are two non-overlapping circles which intersect at points P and Q. R and S are points on ω_1 and ω_2, respectively. If $\angle PRQ = \angle PSQ$, then ω_2 is the reflection of ω_1 in the line PQ.*

Proof. Let r_1 and r_2 be the radius of ω_1 and ω_2, respectively. By generalized law of sine, $r_1 = \frac{PQ}{2\sin\angle PRQ} = \frac{PQ}{2\sin\angle PSQ} = r_2$. ω_1 and ω_2 have the same radii. Also. both circles pass through P and Q. Hence, ω_2 is the reflection of ω_1 in the line PQ. □

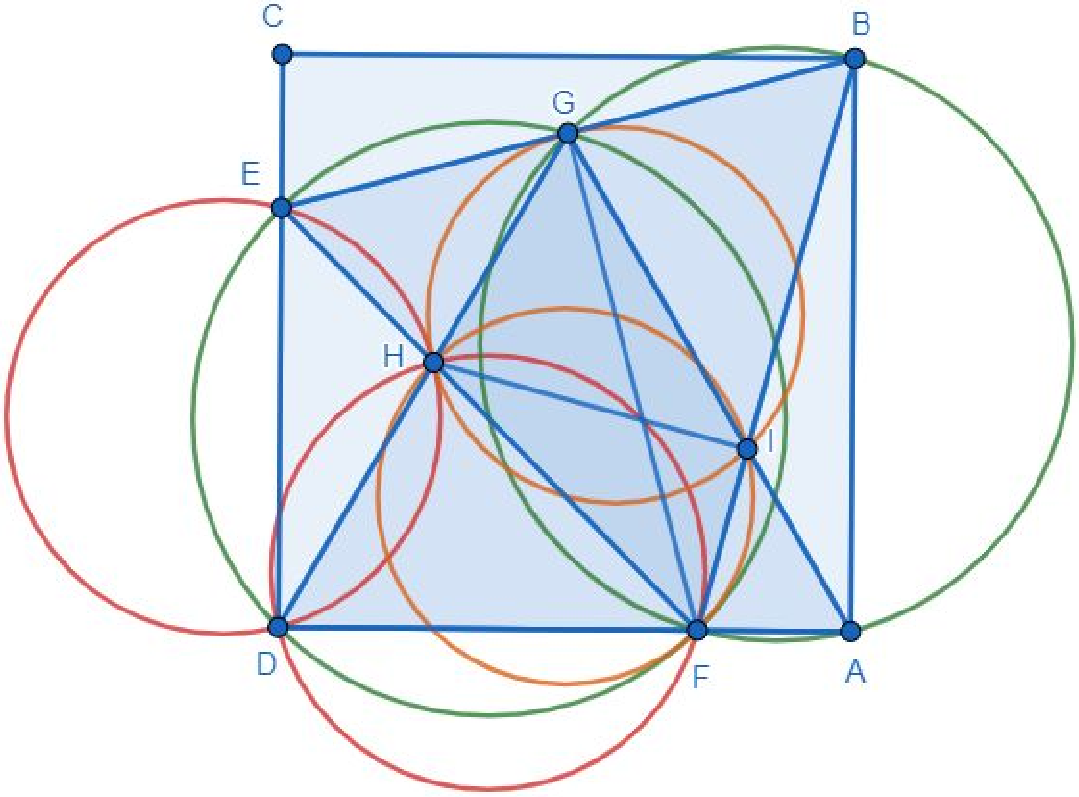

FIGURE 3. Theorem 3.1.

Lemma 3.1 derives the following theorem, where the part (1) is the solution of Problem 1.2 (see Figure 3).

Theorem 3.1. *The following statements are true.*
(1) *The circumcircle of* $\triangle FIH$ *is the reflection of the circumcircle of* $\triangle GHI$ *in the line* HI.
(2) *The circumcircle of* $\triangle DHE$ *is the reflection of the circumcircle of* $\triangle DHF$ *in the line* DH.
(3) *The circumcircle of* $\triangle EFG$ *is the reflection of the circumcircle of* $\triangle AFG$ *in the line* FG.

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