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Further notes on Tamura's problem in the general arbelos

Kai-Uwe Ekrutt[a] and Floor van Lamoen[b] [2]
[a] Veilchenweg 21, 58313 Herdecke, Germany
e-mail: kai-uwe.ekrutt@onlinehome.de
[b] Statenhof 3, 4463 TV Goes, The Netherlands
e-mail: fvanlamoen@planet.nl

Abstract. We further explore the generalization Okumura [2] has given of Tamura's problem to the general arbelos, resulting in the *Tamura twins*. We connect these twins to the famous Archimedean twins. While Okumura found a third circle congruent to the Tamura twins, we present four more of these, calling them *Tamura circles*.

Keywords. arbelos, circle inversion, Archimedean circles, congruent non-Archimedean circles.

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1. Introduction

One of the arbelos problems in Wasan geometry, Tamura's problem, was described by Hiroshi Okumura in [2]. It presents a symmetric arbelos on segment AB with smaller semicircles α and β and larger semicircle γ and a line h perpendicular to AB. The problem now states that if the circle tangent h and to α and β externally and the incircle of the curvilinear triangle formed by β, γ and h are congruent, then their radius is $\frac{1}{10}$ of the base segment. See Figure 1.

Okumura [2] generalized Tamura's problem to the general arbelos. If the arbelos radii are given as a of α, b of β, and $c = a + b$ of γ, then Okumura found that the radii of the congruent circles defined as in Tamura's problem, which we will call *Tamura twins*, are

$$\frac{abc}{a^2 + c^2}. \tag{1}$$

[2]Corresponding author

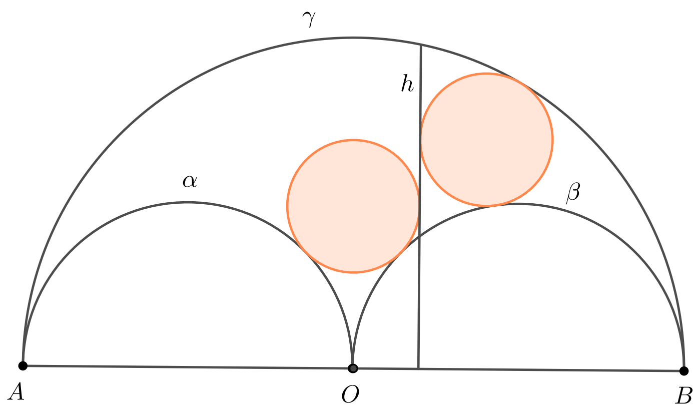

FIGURE 1. Tamura's problem

In addition Okumura found a third circle congruent to the Tamura twins, with the points G and K where h meets β and AB respectively as endpoints of its diameter. We will call circles as congruent to the Tamura twins *Tamura circles*.

We present an alternative way to identifiy the Tamura twins and present a proof that connects these circles directly to the famous Archimedean twins. After having done so we present a few more Tamura circles.

2. THE TAMURA TWINS

In private communication between the authors, the first author proposed the following theorem, for which we will later show that the theorem is about the Tamura twins. The theorem appears to have been published in 1949 by Gertrude Welch [4].

Theorem 2.1. *We have an arbelos on segment AB with smaller semicircles $\alpha = (AO) = M_\alpha(a)$, $\beta = (OB) = M_\beta(b)$ and larger semicircle $\gamma = (AB) = M_\gamma(c)$, hence $c = a + b$. We construct a fourth semicircle $\delta = (AM_\beta)$. The circles ε_1 tangent externally to α and β and internally to δ and ε_2 tangent externally to β and δ and internally to γ are congruent. See Figure 2.*

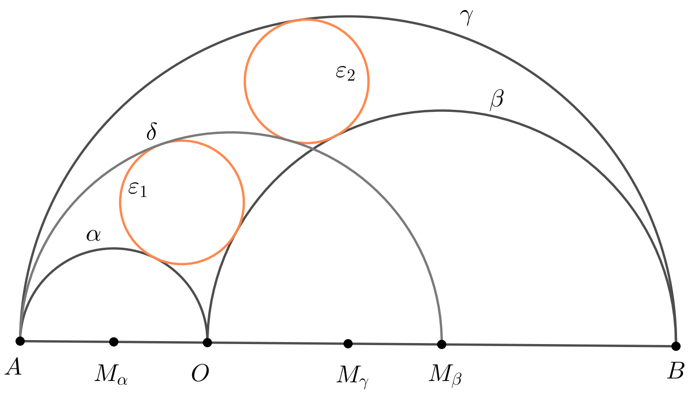

FIGURE 2. Ekrutt's theorem

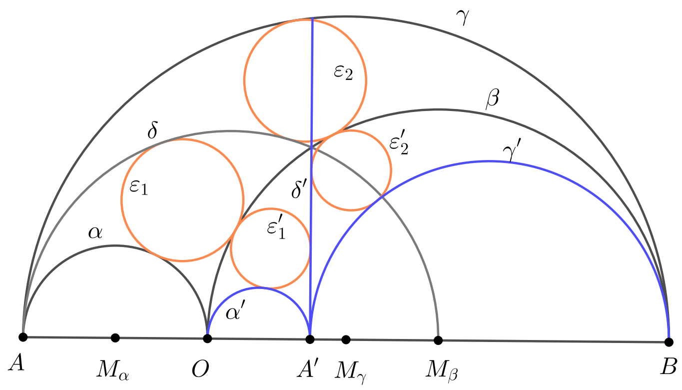

FIGURE 3. The smaller arbelos through inversion $\mathcal{I}$

Proof. Consider inversion $\mathcal{I}$ with inversion circle β. Together with β the images α' of α and γ' of γ form a smaller arbelos. The image δ' of δ is the common tangent of α' and γ' perpendicular to AB. Hence the images ε_1' and ε_2' of ε_1 and ε_2 respectively are exactly the twin circles of Archimedes with respect to this smaller arbelos. It is well known that they are congruent and as they are equally distant from the inversion center M_β, so are their originals ε_1 and ε_2. See Figure 3. $\square$

To establish the radii of ε_1 and ε_2 we start to calculate the radii a' of α' and b' of γ' respectively, the radii of the smaller arbelos' semicircles. Using $\mathcal{I}$ we have $M_\beta A \cdot M_\beta A' = b^2$ and we find $M_\beta A' = \frac{b^2}{a+c}$ and

$$a' = \frac{1}{2}OA' = \frac{ab}{a+c},$$

$$b' = b - a' = \frac{bc}{a+c}.$$

Using the well-known formula for the radius of Archimedean circles we find that the radius r' of ε_1' and ε_2' is given by

$$r' = \frac{a'b'}{a'+b'} = \frac{a'b'}{b} = \frac{abc}{(a+c)^2}.$$

We finally use $\mathcal{I}$ to establish the radius r of ε_1 and ε_2 by

$$(b-2r')(b+2r) = b^2.$$

This indeed yields

$$r = \frac{abc}{a^2+c^2}. \tag{2}$$

Note that the radii of ε_1 and ε_2 can be calculated without inversion as well. Kousik Sett [3] used for instance Stewart's theorem.

We see that (2) is equal to the radius (1) Okumura found for the Tamura twins. Also, ε_1 is tangent to α and β externally and ε_2 is tangent to β externally and γ internally, just as the Tamura twins. This means that ε_1 and ε_2 are indeed the same circles as the Tamura twins, and hence that ε_1 and ε_2 have a common tangent h, perpendicular to AB at a point K.

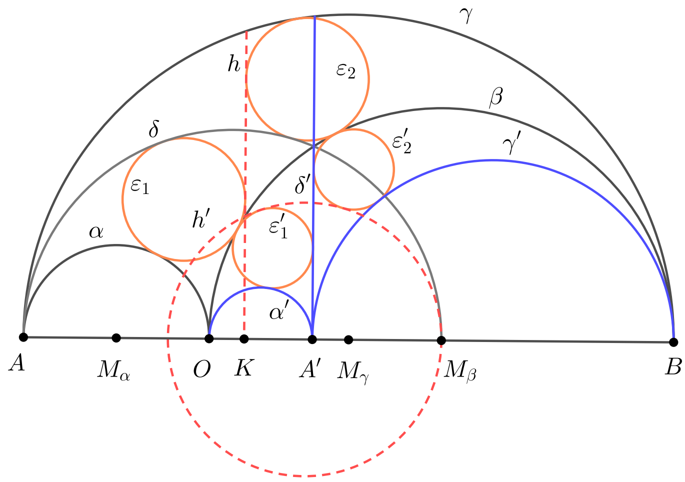

Figure 4. Circle h'

Note that the existence of h can also be proven by calculation, see for instance Kousik Sett's proof [3].

Applying $\mathcal{I}$ to h, we find a circle h'. With respect to the smaller arbelos this circle is tangent to both Archimedean twins, one internally and one externally, has its center on AB, intersecting it at the center of the greater semicircle of this arbelos. The existence of this circle in the arbelos seems not to be well known. See Figure 4.

To further study h' we consider circle Γ with the same properties in the general arbelos. Without losing generality we assume $a < b$ (for $a = b$ the circle degenerates to the common tangent of α and β perpendicular to AB). Γ is tangent to the Archimedean twins, the a-circle internally and the b-circle externally, has its center on AB, and passes through M_γ. We introduce Cartesian coordinates with $O(0, 0)$, $A(-2a, 0)$ and $B(2b, 0)$. The radius of the Archimedean twins is known as (see [1])

$$r_A = \frac{ab}{c}.$$

The centers of the Archimedean twins are $A_a(-r_A, 2\sqrt{ar_A})$ and $A_b(r_A, 2\sqrt{br_A})$. Now Γ is given by

$$\Gamma : \left(x - \frac{a^2 - 4ab + b^2}{2(b-a)}\right)^2 + y^2 = \left(\frac{a^2 + b^2}{2(b-a)}\right)^2,$$

hence has midpoint

$$M_\Gamma \left(\frac{a^2 - 4ab + b^2}{2(b-a)}, 0\right).$$

It is readily checked that indeed

$$\mathrm{d}(A_a, M_\Gamma) = \frac{a^2 + b^2}{2(b-a)} - r_A$$

and

$$\mathrm{d}(A_b, M_\Gamma) = \frac{a^2 + b^2}{2(b-a)} + r_A$$

and that $M_\gamma(b - a, 0)$ lies on Γ.

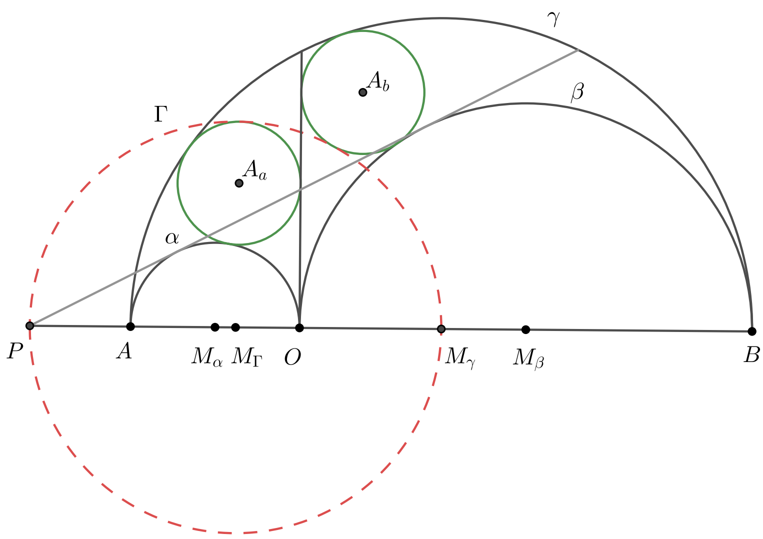

FIGURE 5. Γ and the Archimedean twins in the general arbelos

Interestingly Γ intersects AB (produced) in $P(\frac{2ab}{a-b}, 0)$ as well, the external center of similitude of α and β. See Figure 5.

Going back to the smaller arbelos and h', this means that the common t tangent to α' and γ' apart from δ' has as image by $\mathcal{I}$ a circle ζ that passes through K and M_β and tangent to α and γ.

This opens the way to a fourth Tamura circle, as the largest circle enclosed by t and β is the well-known Archimedean *Bankoff quadruplet circle* [1]. As tangent to β internally its image by $\mathcal{I}$ is a Tamura circle ε_4, the largest circle enclosed by β and ζ, while tangent to β externally. Its point of tangency to β is the point where β meets its tangent u from A and δ and it passes through Q, the point where ζ and h intersect. See Figure 6.

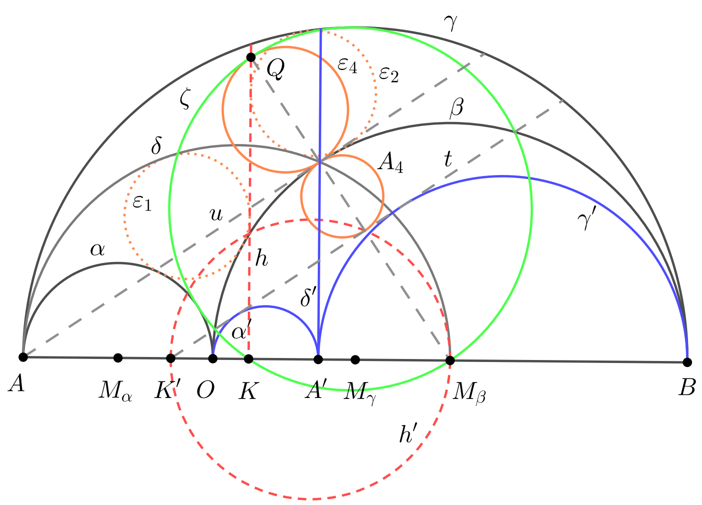

FIGURE 6. The fourth Tamura circle ε_4

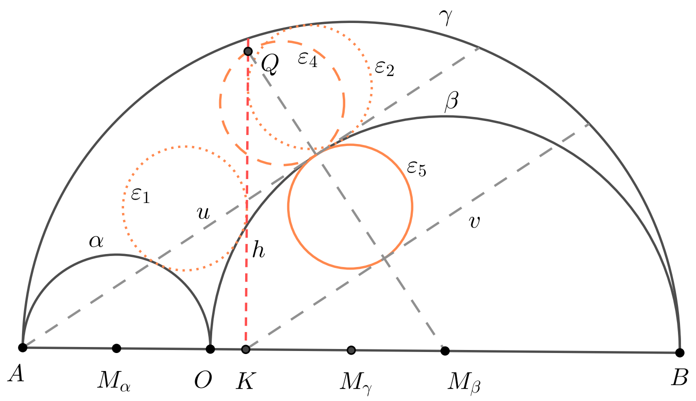

FIGURE 7. The fifth Tamura circle ε_5

3. ARE TAMURA CIRCLES UBIQUITOUS?

Okumura [2] found a third Tamura circle, which we will refer to as ε_3, above we presented a fourth one. It may be more in the spirit of modern computer age - with many mathematical online encyclopedias and catalogues - than in the spirit of Wasan geometry to look for more of those. Nevertheless in this final section we will present a few ones that we found, presenting them to leave details to the reader.

The fifth Tamura circle ε_5 is the reflection of ε_4 through u and is also tangent to the line v through K parallel to u. See Figure 7. Note that $M_\beta Q$ passes through the centers of ε_4 and ε_5 and is perpendicular to t, u and v.

Now, let semicircle $\kappa = (M_\beta B)$, let semicircle δ_1 be the symmetric of δ with respect to the perpendicular through M_γ to AB. Then let M_6 be the midpoint of BQ. The Tamura circle ε_6 with center M_6 is tangent to δ, δ_1, ζ, and κ. The tangent w of ε_6 and κ through their point of tangency passes through M_γ and is parallel to t, u and v.

For the seventh Tamura circle, we first note that one of the endpoints G of the defining diameter of ε_3 lies on the line connecting A with the midpoint R of the arc (OB). Let S be the midpoint of arc $(M_\beta B)$ and M_7 the point where $M_\gamma S$ meets κ apart from S. Then clearly $\mathrm{d}(M_7, AB) = \frac{GK}{2} = r$ and M_6M_7 is perpendicular

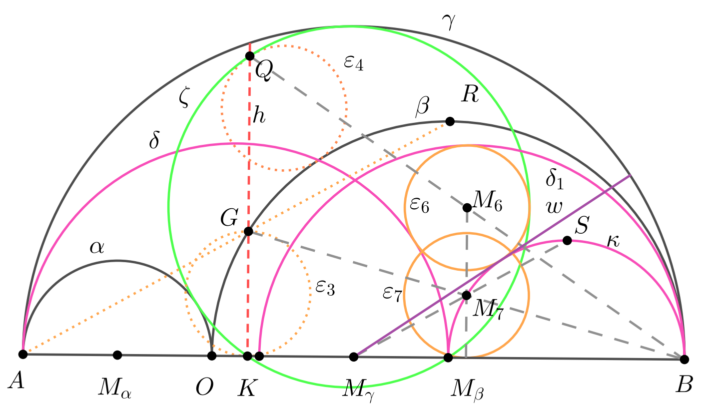

FIGURE 8. The sixth and seventh Tamura circles ε_6 and ε_7

to AB. So the Tamura circle ε_7 with center M_7 is tangent to AB. See Figure 8. Note that $AR \parallel M_\gamma S$.

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