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On a Sangaku like configuration involving a triangle and three congruent circles

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Abstract. We present some geometric conditions for the existence of solutions of a sangaku like configuration involving three equal circles, each touching two sides of a triangle. In a limit case naturally appearing in this study, the configuration reduces to the well-known Sangaku from the Chiba prefecture.

Keywords. triangle geometry, Sangaku.

Mathematics Subject Classification (2020). 51-02, 51M15.

1. Introduction

The present discussion was triggered from a question addressed to me about a Sangaku-like problem. [2] It was about the configuration of Figure 1.

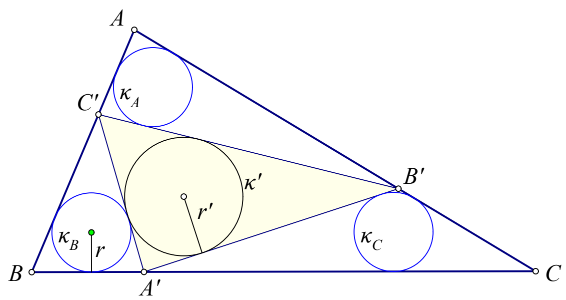

Figure 1. A Sangaku-like problem

In this the triangle $A'B'C'$ is inscribed in the triangle ABC and is simultaneously externally tangent to three equal circles κ_A, κ_B and κ_C of radius r, each in turn

[2]I am greatly indebted to the referee for many detailed suggestions that contributed to a better presentation of this material.

tangent to two sides of ABC and each tangent to a different side of $A'B'C'$. It was to prove that $r + r' = \rho$, where ρ is the inradius of ABC.

In fact, it is not difficult to show that the inscribed triangle $A'B'C'$ has the same area and perimeter with the one formed by the centers of the three equal circles (see Figure 2), from which the result follows immediately.

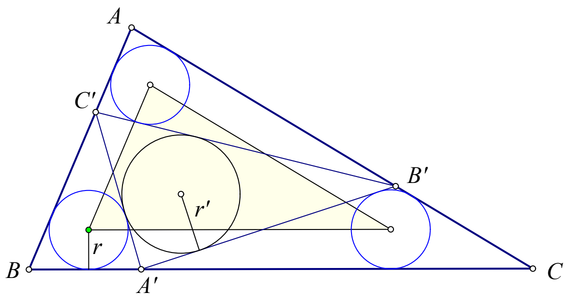

Figure 2. Two triangles with equal areas and perimeters

I refer to this, which could be well known, as *"the problem"*. For the completeness of exposition I supply a proof in the next section. My interest however focused on two other questions: (i) How big can be the radius r and have still a triangle $A'B'C'$ as in this configuration? (ii) How many such triangles $A'B'C'$ exist for a given r that allows such an existence? The investigation of some limits for the existence of solutions to this problem led in a natural way to the well-known Sangaku from the Chiba prefecture.

Regarding the organization of the article, in section 2 we supply a short proof of *the problem*. In section 3 we determine the *"admissible configurations"* i.e. those for which we can find a solution to *the problem*. In section 4 we study an ellipse intimately connected with the existence of solutions. In section 5 we draw some further restrictions on the existence of solutions in connection with the aforementioned ellipse. In section 6 we study the width of a certain strip containing this ellipse. Finally, in section 7 we discuss some bounds for the existence of solutions and a resulting limit configuration coinciding with that of the Sangaku from the Chiba prefecture.

2. Solution of the initial problem

Comparing the tangents from the points B', C' and A' we see that the perimeters of the two triangles $A'B'C'$ and $A_0B_0C_0$ are equal (see Figure 3).

For the computation of areas, we notice first that the three quadrangles at the vertices of ABC: $AA_1A_0A_3$, $BB_1B_0B_3$ and $CC_1C_0C_3$, glued together, create a triangle $A''B''C''$ similar to ABC (see Figure 4) with an incircle or radius r. Hence the similarity ratio is r/ρ. Denoting the semi-perimeters of the triangles $ABC, A'B'C', A_0B_0C_0$ and $A''B''C''$ correspondingly by τ, τ', τ_0 and τ'' and the corresponding areas $E = (ABC)$ and $E' = (A'B'C'), \ldots$, we have, using the formulas $E = \rho\tau$, $E' = r'\tau', \ldots$:

$$E' = E - (AC'B') - (BA'C') - (CB'A') = \rho \cdot \tau - r \cdot (\tau + \tau') .$$

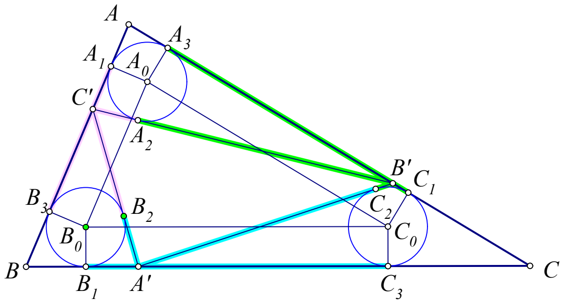

Figure 3. Two triangles with equal perimeters

But we saw that $\tau' = \tau_0$ and $A_0B_0C_0$ is similar to ABC in ratio $\frac{\rho - r}{\rho}$. Hence the semi-perimeter τ' of $A'B'C'$ is

$$\tau' = \frac{(\rho - r)}{\rho}\tau \ .$$

On the other hand, the area $E_0 = (A_0B_0C_0)$ results by subtracting from the area $E = (ABC)$ the areas of the three rectangles and the area $E'' = (A''B''C'')$:

$$E_0 = E - 2r\tau_0 - \frac{r^2}{\rho^2}E = \rho\tau - 2r\frac{\rho - r}{\rho}\tau - \frac{r^2}{\rho^2}\rho\tau = \tau\frac{(\rho - r)^2}{\rho} \ .$$

The claim $\rho = r + r'$ results obviously from $r' = E'/\tau' = E_0/\tau_0 = \rho - r$.

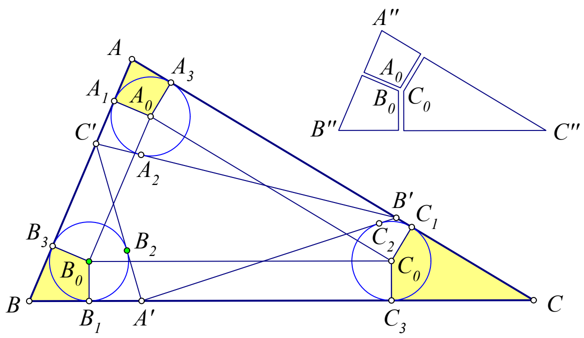

Figure 4. Two triangles with equal areas

3. The admissible configuration

Let us assume that the circles κ_B and κ_C , tangent to the side BC, are sufficiently small. Let also ζ_B and ζ_C be the tangents respectively from A to κ_B and κ_C different from the sides AB and AC. Let finally Y_1 and Y_2 be their intersection with BC (see Figure 5-(I)).

We see easily, that if there is a chance to find a solution to *the problem*, the vertex A' of the inscribed triangle $A'B'C'$ has to be in the interval $[Y_1, Y_2]$. The width of this interval decreases as the equal circles κ_B and κ_C become bigger and bigger, until their radius reaches a critical value, for which the two tangents ζ_B and ζ_C coincide with a common tangent ζ_0 to the two circles from A. The circles defined for this critical value and their common tangent ζ_0 through A define the so-called *"Sangaku from the Chiba prefecture"* ([3] , [4]) (see Figure 5-(II)).

For this limit configuration there is obviously no solution to our problem. Hence the radius r of the circles has to be less than this critical radius of the corresponding Sangaku circles associated to the side BC of the triangle. Since the same reasoning is valid for any side of the triangle we conclude, that the radius r of the

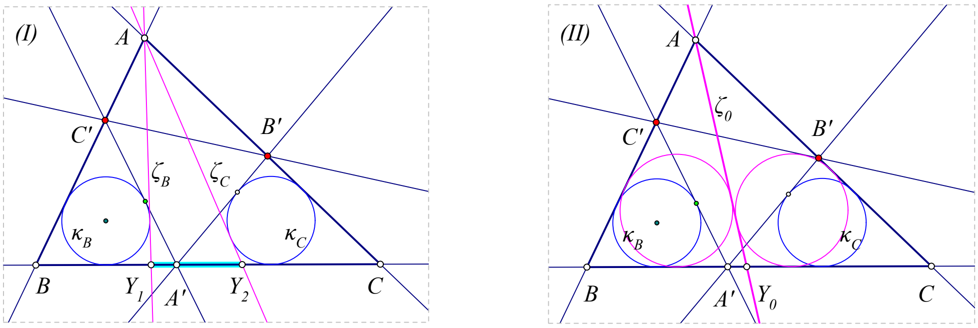

FIGURE 5. Acceptable domain of location of A'

three circles has to be less than the radius of the Sangaku circles corresponding to the smallest side of the triangle ABC. We call a configuration *"admissible"* if it satisfies this restriction. Subsequently we'll deal with admissible configurations and we'll determine also the critical value of r related to the Sangaku from Chiba.

4. An elliptic envelope

The following procedure seems to be a natural way to search for an inscribed triangle touching the three circles (see Figure 6-(I)): Select a side, BC say, and move a point Y on it, drawing the tangents YX and YZ to the two circles

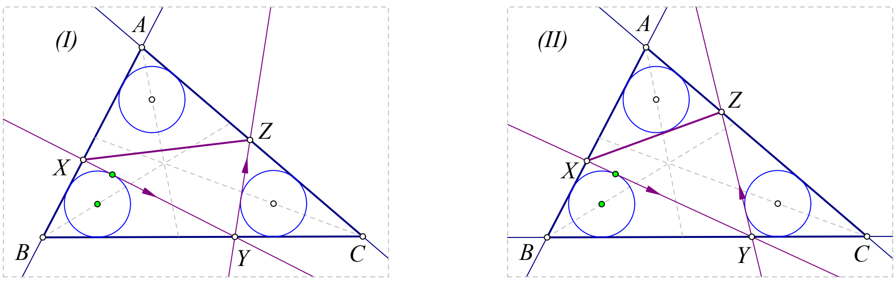

FIGURE 6. Positions of XZ for varying $Y \in BC$

tangent to it and intersecting the other sides at X and Z. As Y moves on BC it may happen that the segment XZ becomes tangent to the third circle (see Figure 6-(II)). In fact, it is well known that the map $f_1 : X \mapsto Y$ of the line AB to line BC, defined by the variable tangent to the circle that touches AB and AC, is a homography [2, §11]. Similarly the map $f_2 : Y \mapsto Z$ of the line BC to line CA is a homography, hence their composition $f = f_2 \circ f_1 : X \mapsto Z$ is a homography of the line AB to line AC. By a well known theorem ([1, p.6]), the line XZ envelopes a conic κ tangent to the lines AB and AC. Also we see easily that for special positions of Y the line XZ takes the position of the three common tangents to the two circles, which are different from the line BC (see Figure 7-(I)). The tangent parallel to BC is attained when Y is at infinity and

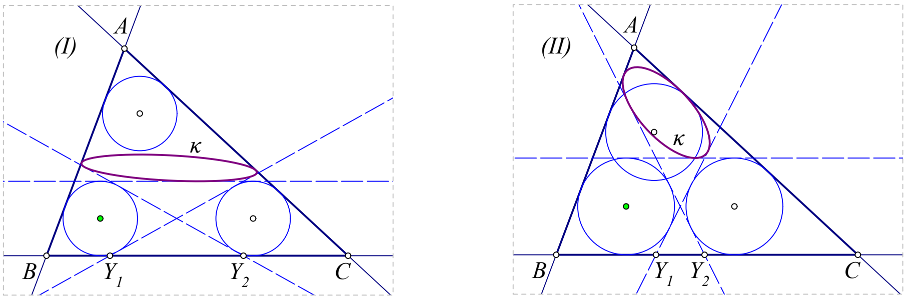

FIGURE 7. Ellipse enveloping the lines XZ for $Y \in BC$

the other two are the inner tangents to the circles, defining their intersections Y_1 and Y_2 with the line BC.

This implies that the conic κ is easily constructed as a tangent to five known lines. The shape of the conic varies with the radius r of the circles. The nature however of the wanted inscribed triangle XYZ, as we remarked in the preceding section, does not allow that two of the three equal circles intersect as this is seen in Figure 7-(II).

Theorem 4.1. *With the preceding notation and conventions, if the configuration is admissible, then the conic enveloping the lines $\varepsilon_Y = XZ$ is an ellipse contained in the triangle ABC.*

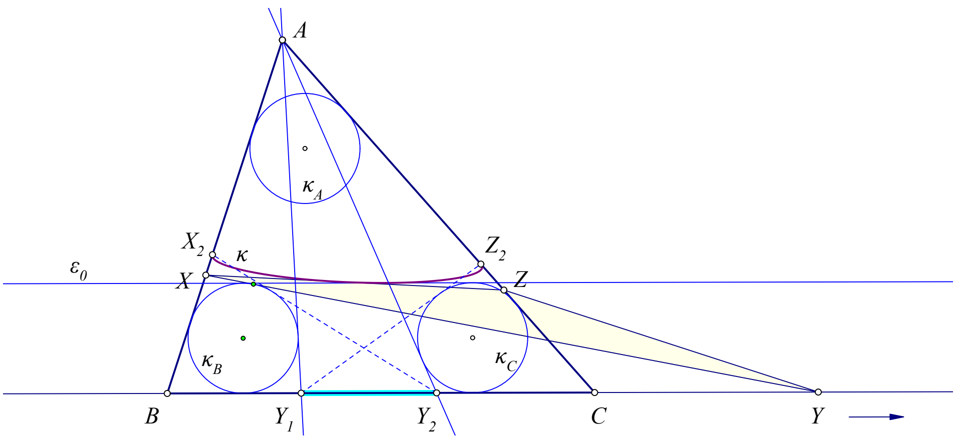

FIGURE 8. The monotonic increase of the slope of $\varepsilon_Y = XZ$ for $Y \to \infty$

Proof. We consider the interval $[Y_1, Y_2] \subset [B, C]$ defined by the tangents AY_1 and AY_2 from vertex A, as in Figure 8. We notice then that the tangents $\varepsilon_Y = XZ$ for Y varying outside this interval have a monotonic behavior.

In fact, for $Y = Y_2$ the tangent $\varepsilon_Y = XZ$ coincides with side AB having corresponding $X = X_2$ and $Z = A$, where X_2 is the contact point of the conic with side AB lying also on a tangent to κ_B from Y_2. As point Y moves on the right of $[Y_1, Y_2]$ to infinity, starting from Y_2, the slope of the tangent $\varepsilon_Y = XY$ is strictly increasing from that of the line AB to the slope of the common tangent ε_0 of the circles κ_B and κ_C which is parallel to BC and is attained for Y at infinity. As Y comes back from infinity to Y_1 from the left, the slope of the tangent ε_Y continues to increase and at $Y = Y_1$ the tangent $\varepsilon_Y = XZ$ has $X = A$ and Z

coinciding with the contact point Z_2 of the conic with side AC lying also on a tangent to κ_C from Y_1. This implies that the *lower* arc (X_2Z_2) of the conic, touching the common tangent ε_0 of the circles κ_B and κ_C which is parallel to BC, is convex towards BC. Thus, from the point A we have two tangents to the conic defining an arc (X_2Z_2) concave towards A. The proof follows from the fact that this behavior can occur only for ellipses. □

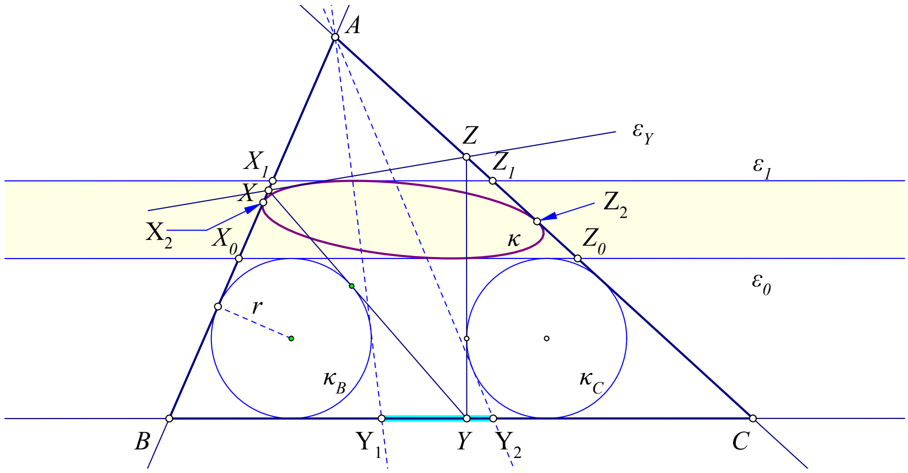

Figure 9. The strip containing the ellipse

Corollary 4.1. *There is a second tangent ε_1 of the ellipse enveloping the lines $\varepsilon_Y = XZ$, parallel to line BC, and the ellipse lies on the domain defined by the intersection of the triangular domain ABC with the strip defined by the two tangents ε_0 and ε_1 which are parallel to the triangle's side BC (see Figure 9).*

For points $Y \in BC$ lying outside the interval $[Y_1, Y_2]$ we see easily that the corresponding tangents $\varepsilon_Y = XZ$ touching, as noticed in the proof of the theorem, the lower arc (X_2Z_2) of the ellipse, intersect one or the other of the two circles κ_B and κ_C. Thus they cannot deliver a solution to *"the problem"* at hand, and we have the following corollary.

Corollary 4.2. *If there is a solution to the problem, then the corresponding tangent $\varepsilon_Y = XZ$ must touch the upper arc (X_2Z_2) of the ellipse and correspond to a point lying in the interval $Y \in [Y_1, Y_2]$.*

Corollary 4.3. *The contact point N of the conic κ with the external common tangent $\varepsilon_0 = X_0Y_0$ of the two circles κ_B and κ_C lies on the Nagel Cevian passing through the vertex A of the triangle ABC (see Figure 10).*

Proof. The proof follows directly from the definition of the contact point as the intersection of two infinitely near lying tangents. The tangent $\varepsilon_0 = X_0Z_0$ to the conic is obtained when the point Y, defining the tangent $\varepsilon_Y = XZ$, is at infinity. Consider then a point Y approaching the point at infinity of the line BC and the corresponding tangent ε_Y of κ. Let T be the contact point of κ with ε_Y, I be the intersection $\varepsilon_0 \cap \varepsilon_Y$, N be the intersection of ε_0 with the Nagel Cevian, and T' be the contact point of the excircle λ in the angle $\widehat{A}$ of the triangle AXZ. As Y tends to the point at infinity of BC, the points T and I tend to coincide with the contact point of ε_0 with κ and the points I and T' tend to coincide with N. Thus, the four points T, I, N and T' at the limit coincide with N. □

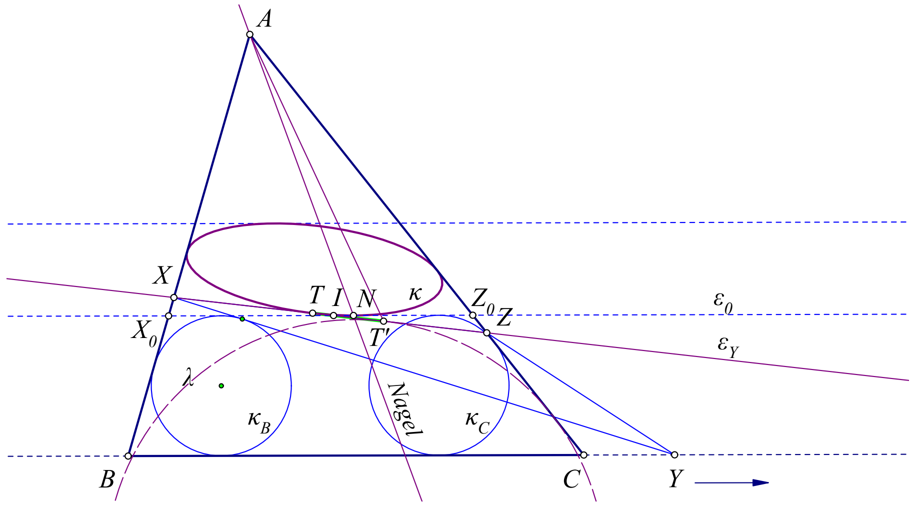

FIGURE 10. The contact point N on the Nagel Cevian from A

Corollary 4.4. *The second tangent $\varepsilon_1 = X_1Z_1$ to the conic κ parallel to BC has its contact point M on the Gergonne Cevian through the vertex A of the triangle (see Figure 11).*

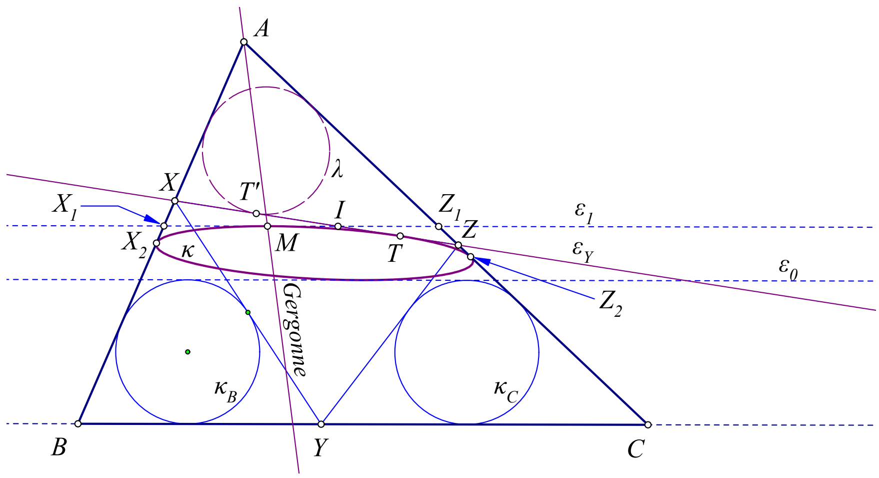

FIGURE 11. The contact point M on the Gergonne Cevian from A

Proof. The proof follows by an argument analogous to the one of corollary 4.3, by considering the coincidence of points T, I and I, T' with M as the variable tangent tends to coincide with ε_1. Here again, T is the contact point of the conic with the variable tangent ε_Y , $I = \varepsilon_y \cap \varepsilon_1$, and T' is the contact point with ε_Y of the incircle λ of the triangle AXZ . □

5. FURTHER RESTRICTIONS FOR THE SOLUTION

Lemma 5.1. *There can be no solution of the problem with the line $\varepsilon_Y = XZ$ tangent to the upper arc (X_AZ_A) of the circle κ_A (see Figure 12-(I)).*

Proof. From the Figure, we see that the lines XY and ZY must not intersect the circle κ_A . This condition is violated when ε_Y touches the upper arc. □

Combining this with corollary 4.2 we conclude that the common tangent ε_Y of the upper circle κ_A and the ellipse of an acceptable solution must separate the circle and the ellipse κ having κ_A tangent at a point of its lower arc $(X_A Z_A)$

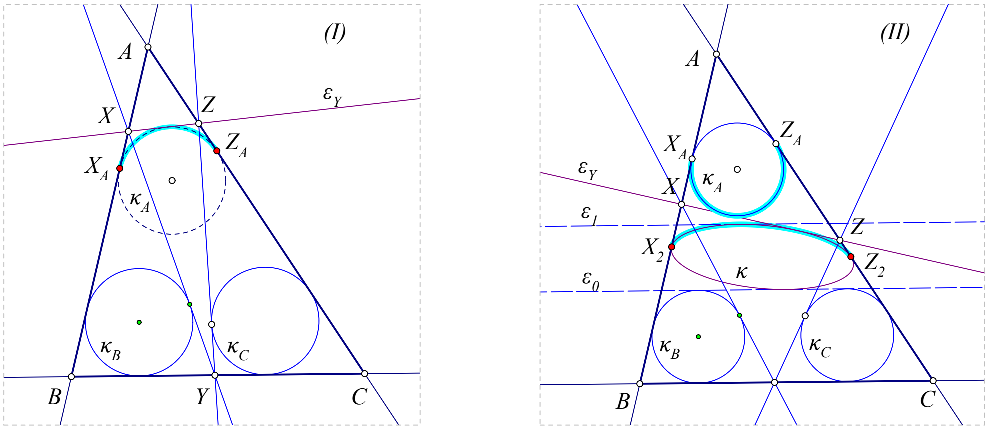

FIGURE 12. Not acceptable and acceptable location of ε_Y

and the conic κ tangent at a point of its upper arc $(X_2 Z_2)$ (see Figure 12-(II)). Next lemmata formulate the possible configurations that may arise in a solution of the problem.

Lemma 5.2. *The problem has no solutions, if the circle* κ_A *intersects the tangent* ε_1 *at two points (see Figure 13-(I)) or lies entirely below* ε_1 *(see Figure 13-(II)).*

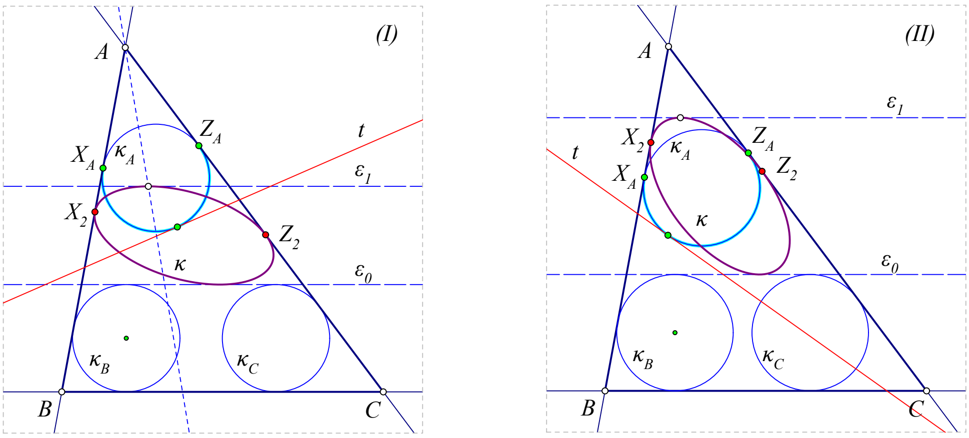

FIGURE 13. Configurations not allowing solutions to *the problem*

Proof. If fact, it is not difficult to see geometrically that in both cases the tangents t to the lower arc $(X_A Z_A)$ of the circle either intersect the ellipse or they do not produce a common tangent with the conic separating the lower arc $(X_A Z_A)$ of the circle and the upper arc $(X_2 Z_2)$ of the conic. □

Lemma 5.3. *If the circle* κ_A *does not intersect the upper tangent* ε_1 *of the conic* κ *lying above it, then there exist two solutions of the problem. If the circle* κ_A *touches* ε_1 *lying above it, then there is precisely one solution represented by the triangle* XYZ *and having* ABC *as its anticomplementary, i.e.* X, Y *and* Z *are the midpoints of the sides of triangle* ABC.

Proof. If the circle κ_A does not intersect ε_1, then it does not intersect also the conic lying below ε_1 (see Figure 14-(I)). We have then precisely two inner common tangents of the circle and the ellipse delivering, each, a solution of the problem.

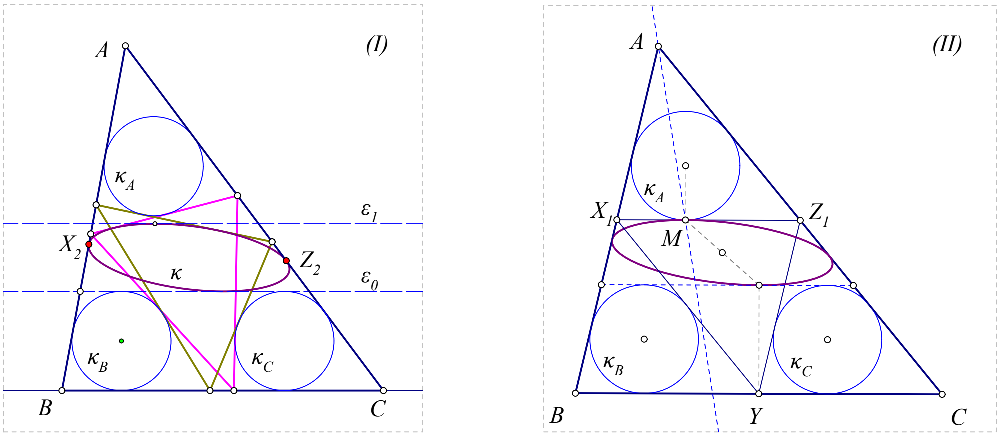

Figure 14. Configurations of precisely two and one only solution

In case the circle κ_A touches the upper tangent $\varepsilon_1 = X_1Z_1$ of the ellipse κ, then its contact point is on a Gergonne Cevian of the triangle AX_1Z_1 (see Figure 14-(II)). From the homothecy of triangles AX_1Z_1 and ABC, we see that the contact point is also on the Gergonne Cevian of ABC. From corollary 4.4 follows then that the contact points of κ_A and κ with ε_1 coincide with the intersection M of the Gergonne Cevian of ABC from A with line ε_1 and the triangle X_1YZ_1 represents then the unique solution of the problem.

There is also a kind of symmetry in the configuration implying that the other sides YX_1 and YZ_1, under the assumption of a unique solution, must be also parallel to corresponding sides of the triangle. In fact, if they were not, and YZ_1, say, was not parallel to AB, then, from the preceding discussion, considering a variable point X_1 on AB we would obtain two acceptable places for X_1 delivering two solutions, which would contradict our assumption. Thus, the sides of ABC are parallel to corresponding sides of X_1YZ_1 thereby proving the lemma. □

6. The width of the strip

In this section we examine the width of the strip containing the ellipse κ of an admissible configuration and its dependence on the radius r of the three circles (see Figure 15). The parallel $\varepsilon_0 = X_0Z_0$ nearer to BC is obviously at distance $2r$ from it. To determine the distance of $\varepsilon_1 = X_1Z_1$ from BC we use the formula expressing the coordinate y of $Y \in BC$ in terms of the coordinate x of $X \in BA$. These coordinates measure the signed distance from B. Thus, $x(B) = y(B) = 0$, $x(A) = c = |AB|$ and $y(C) = a = |BC|$. We use also the analogous coordinate z along line CA with $z(C) = 0$ and $z(A) = b = |CA|$. The well known formula ([2, p.13]) uses also the distance d_B of the vertex B from the contact points of the sides with the circle κ_B:

$$y = f(x) = (d_B^2 + r^2)\frac{x - d_B}{d_B x - (d_B^2 + r^2)}. \tag{1}$$

Using the analogous formula and notation to express z in terms of y we have

$$(2) \qquad z \;=\; g(y) \;=\; (d_C^2+r^2)\frac{(a-y)-d_C}{d_C(a-y)-(d_C^2+r^2)} \;.$$

The composition is

$$(3) \qquad z \;=\; h(x) \;=\; g(f(x)) \;=$$

$$(r^2+d_C^2)\frac{(r^2+d_B(d_B+d_C-a))x-(d_B+d_C-a)(r^2+d_B^2)}{(d_B+d_C)r^2+d_Bd_C(d_B+d_C-a))x-(r^2+d_B^2)(r^2+d_C(d_B+d_C-a)} \;.$$

We use this formula to compute the values of x for which the line $\varepsilon_Y = XZ$ is horizontal. This, in terms of coordinates, translates to equation

$$(4) \qquad \frac{z}{x} \;=\; \frac{b}{c} \;,$$

and leads to a quadratic equation in x with seemingly complicated coefficients.

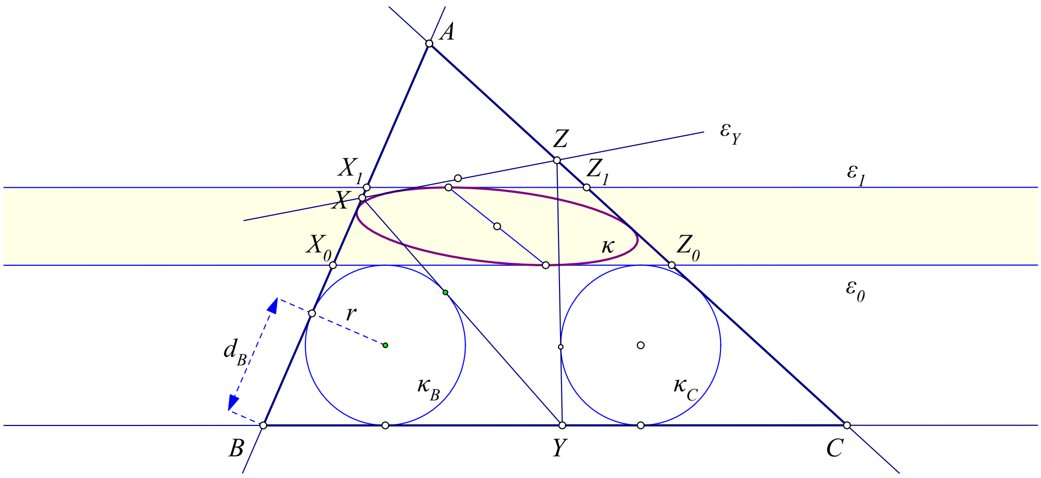

Figure 15. The strip containing the ellipse

There are however relations leading to considerable simplification. The obvious one suggested by Figure 15 and studied in [5, p.8] is:

$$\frac{r}{d_B} \;=\; \tan\left(\frac{\widehat{B}}{2}\right) \;=\; \sqrt{\frac{(\tau-a)(\tau-c)}{\tau(\tau-b)}} \qquad \Rightarrow \qquad r^2 = d_B^2\frac{(\tau-a)(\tau-c)}{\tau(\tau-b)} \;,$$

which allows the elimination of r^2 from equation (3). Then, combining the last formula with the corresponding

$$\frac{r}{d_C} \;=\; \tan\left(\frac{\widehat{C}}{2}\right) \;=\; \sqrt{\frac{(\tau-a)(\tau-b)}{\tau(\tau-c)}} \qquad \Rightarrow \qquad d_C = d_B\frac{\tau-c}{\tau-b} \;,$$

we obtain from equations (3) and (4) a quadratic equation in x, whose coefficients can be expressed using only d_B and the side-lengths a, b and c of the triangle of reference. In fact, dropping the calculation and coming to the end result, we see that the quadratic equation splits, as expected, into two linear equations

$$(5) \qquad \tau(\tau-b)x - d_Bac \;=\; 0 \;,$$

$$(6) \qquad (\tau-b)(\tau(\tau-b)-(b+c)d_B)x - ac(\tau-b-d_B)d_B \;=\; 0 \;.$$

Denoting by x_0 and x_1 the solutions of the first and second equation we come to the expression involving the semi-perimeter τ and the inradius ρ of the triangle of reference

$$\frac{x_0}{x_1} = 1 - \frac{r}{\rho - r} \cdot \frac{\tau - a}{\tau}. \tag{7}$$

As we noticed, x_0 is the value of x determining the lower parallel to BC, which is a common tangent to the circles κ_B and κ_C. In fact, it can be easily verified that

$$x_0 = \frac{2r}{\sin(\widehat{B})} = \frac{rac}{\tau\rho}, \quad x_1 = \frac{acr(\rho - r)}{\rho(\rho\tau - 2r\tau + ar)}, \tag{8}$$

$$x_1 - x_0 = \frac{ac(\tau - a)}{\rho\tau} \cdot \frac{r^2}{\rho\tau - 2r\tau + ar}, \tag{9}$$

which, taking into account that $\sin(\widehat{B}) = 2\rho\tau/(ac)$ leads to the expression for the width of the strip

$$w = \sin(\widehat{B})(x_1 - x_0) = \frac{2(\tau - a)r^2}{\rho\tau - (2\tau - a)r}. \tag{10}$$

7. Bounds and limits

The value of x_1 increases from 0 to ∞ as r varies from 0 to the limit of r expressed by $r_+ = (\rho\tau)/(b + c)$, after which it ceases to be positive (see Figure 16). From the preceding discussion we have that the existence of solutions occurs

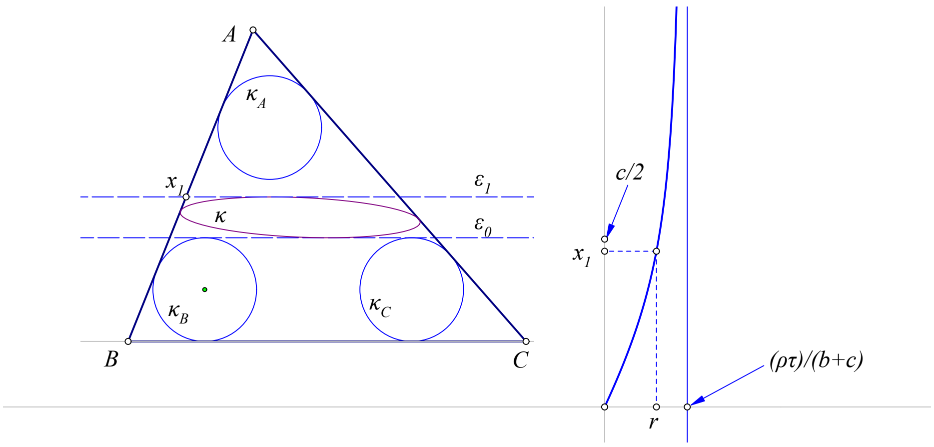

Figure 16. The function $x_1(r)$

in the half-closed interval $r \in (0, \rho/2]$. For values of the radius $r > \rho/2$ we have no solution to the problem but the strip and the conic contained in it continue to exist at least as long as $x_1 < c$. Figure 17 shows some of the enveloping ellipses for values of $r < r_0$ later being the critical radius for which $x_1(r_0) = c$. Using equation (8) we obtain for the corresponding r_0 and x_0 the values

$$r_0 = \frac{\rho}{a}\left(\tau - \sqrt{\tau(\tau - a)}\right) \quad \text{and} \quad x_0 = c\left(1 - \sqrt{\frac{\tau - a}{\tau}}\right). \tag{11}$$

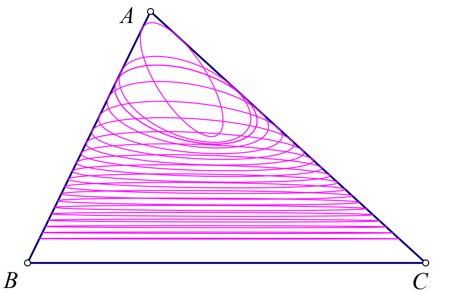

Figure 17. Some enveloping ellipses for values of $r < r_0$

Figure 18-(I) shows a typical instance of an ellipse κ as x_1 tends to coincide with c corresponding to the point A. The contact points N, M of the ellipse with lines $\varepsilon_0, \varepsilon_1$ move respectively on the Nagel and Gergonne Cevians from A.

As x_1 tends to A, the internal tangents η_B and η_C of the circles κ_B and κ_C tend to the internal tangents ζ_A and ξ_A of the corresponding limit circles κ_B and κ_C (see Figure 18-(II)). Also point N tends to the intersection N_0 of the Nagel Cevian with the corresponding line ε_0. The ellipse κ tends to coincide with the

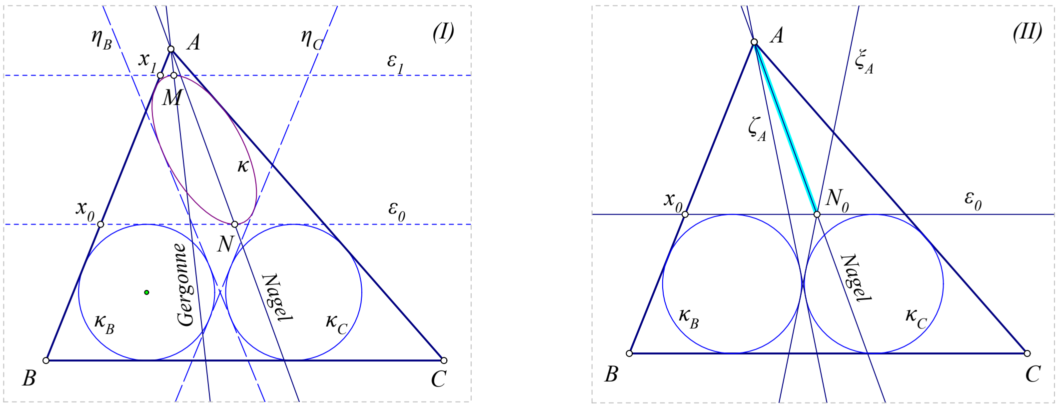

Figure 18. Sangaku from Chiba prefecture as a limit case

segment AN_0 and the whole limit configuration is the one of the Sangaku from Chiba, with ζ_A the common tangent from A of the two equal circles. We notice that ξ_A is the symmetric of ζ_A with respect to the line of centers of the circles κ_B, κ_C and passes through N_0.

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