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Properties of Ajima Circles

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Abstract. We study properties of certain circles associated with a triangle. Each circle is inside the triangle, tangent to two sides of the triangle, and externally tangent to the arc of a circle erected internally on the third side.

Keywords. Apollonius circle, Gergonne point, tangent circles.

Mathematics Subject Classification (2020). 51-02, 51M04.

1. Introduction

The following figure appears in a Sangaku described in [14] and reprinted in [21].

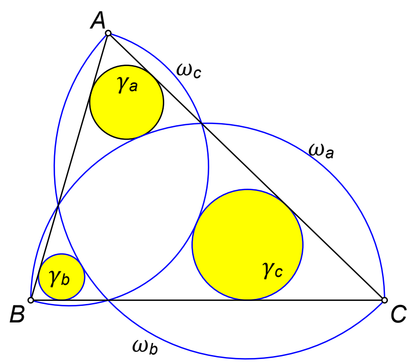

Figure 1. Sangaku configuration

In this figure, the semicircle erected inwardly on side BC is named ω_a. Semicircles ω_b and ω_c are defined similarly. The circle inside $\triangle ABC$, tangent to sides AB and AC, and externally tangent to semicircle ω_a is named γ_a. Circles γ_b and γ_c

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are defined similarly. The sangaku gave a relationship involving the radii of the three circles.

Additional properties of this configuration were given in [29] and [30]. For example, in Figure 2 (left), the three blue common tangents are all congruent. Their common length is $2r$, twice the inradius of $\triangle ABC$. In Figure 2 (right), the six touch points lie on a circle with center I, the incenter of $\triangle ABC$.

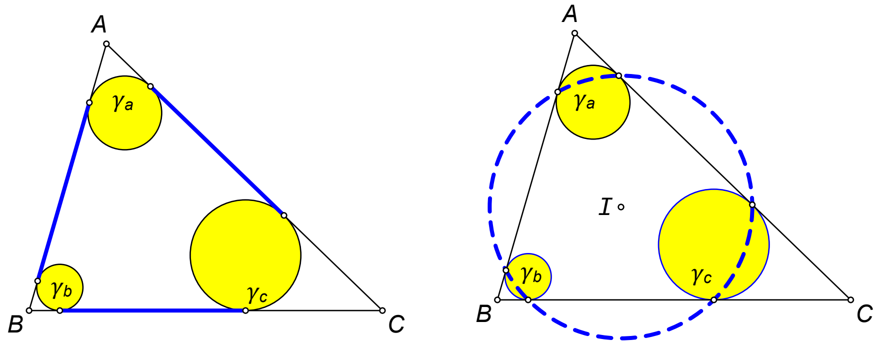


FIGURE 2. properties

It is the purpose of this paper to present properties of circles such as γ_a and also to generalize these results by replacing the semicircles with arcs having the same angular measure.

2. PROPERTIES OF ω_a AND γ_a

In this section we will discuss properties of the configuration shown in Figure 3 in which ω_a is any circle passing through vertices B and C of $\triangle ABC$. The circle γ_a is inside $\triangle ABC$, tangent to sides AB and AC and tangent to ω_a at T. This circle is sometimes known in the literature as an Ajima circle [10].

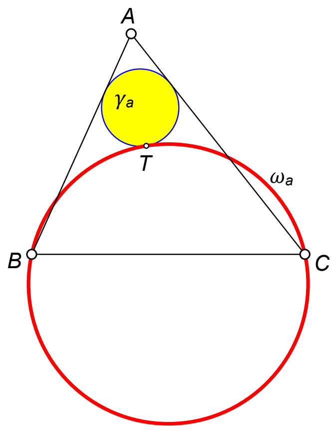


FIGURE 3. The configuration we are studying

An *Ajima circle* of a triangle is a circle (γ) that is tangent to two sides of the triangle and also tangent to a circle (ω) passing through the endpoints of the third side. In this paper, we are primarily interested in Ajima circles that lie inside the triangle and for which γ and ω are *externally* tangent as shown in Figure 3.

Occasionally, we will generalize a result and present a theorem in which γ_a is any circle tangent to AB and AC (not necessarily tangent to ω_a). To help the reader recognize when a result applies to an Ajima circle, we will color all Ajima circles yellow.

The standard notation related to our configuration that we use throughout this paper is shown in the following table.

Standard Notation	
Symbol	**Description**
a, b, c	lengths of sides of $\triangle ABC$
ω_a	circle through points B and C
γ_a	Ajima circle inscribed in $\angle BAC$ tangent to ω_a
D	center of γ_a
T	Unless specified otherwise, T is the point where γ_a touches ω_a.
O_a	center of ω_a
I	incenter of $\triangle ABC$
r	inradius of $\triangle ABC$
R	circumradius of $\triangle ABC$
p	semiperimeter of $\triangle ABC = (a+b+c)/2$
Δ	area of $\triangle ABC$
S	twice the area of $\triangle ABC$ (i.e. 2Δ)
G_e	Gergonne point of $\triangle ABC$
θ	angular measure of arc $\overset{\frown}{BTC}$

Without loss of generality, we will assume $AB < AC$.

We will survey some known results and give some new properties of this configuration.

The following result is due to Protasov [23]. Proofs can be found in [3] and [1, pp. 90–94].

Theorem 2.1 (Protasov's Theorem)**.** *The segment TI bisects $\angle BTC$ (Figure 4).*

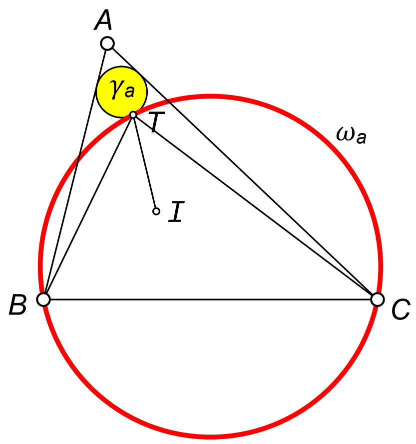


FIGURE 4. TI bisects $\angle BTC$

The following result comes from [34].

Lemma 2.2. *Let Γ and Ω be two circles that are externally tangent at T. Let B and C be points on Ω and let BU and CV be tangents to Γ as shown in Figure 5. Then*

$$\frac{BU}{CV} = \frac{BT}{CT}.$$

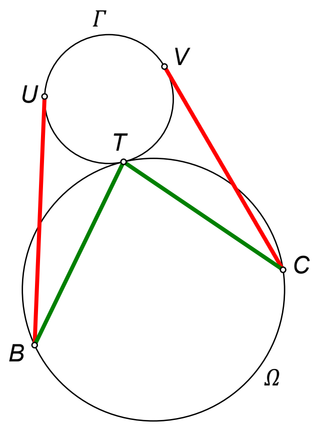


FIGURE 5. $BU/CV = BT/CT$

Proof. Let BT meet Γ at Q and let CT meet Γ at P. Let XY be the tangent to both circles at T (Figure 6).

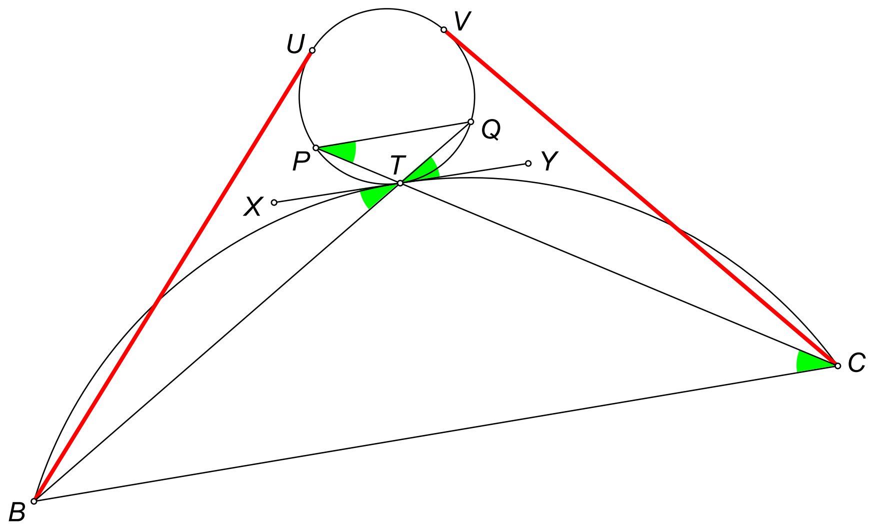


FIGURE 6.

We have

$$\angle TPQ = \frac{1}{2}\widehat{QT} = \angle YTQ = \angle XTB = \frac{1}{2}\widehat{BT} = \angle TCB.$$

Since $\angle QTP = \angle BTC$, we find that $\triangle PQT \sim \triangle CBT$. Thus,

$$\frac{BT}{QT} = \frac{CT}{PT}$$

which implies

$$\frac{BT}{BQ} = \frac{CT}{CP} \quad \text{or} \quad \frac{BQ}{CP} = \frac{BT}{CT}.$$

Since BU and CV are tangents, we have $(BU)^2 = BT \cdot BQ$ and $(CV)^2 = CT \cdot CP$. Combining, we get

$$\frac{(BU)^2}{(CV)^2} = \frac{BT \cdot BQ}{CT \cdot CP} = \left(\frac{BT}{CT}\right)\left(\frac{BQ}{CP}\right) = \left(\frac{BT}{CT}\right)\left(\frac{BT}{CT}\right) = \frac{(BT)^2}{(CT)^2}$$

which implies $BU/CV = BT/CT$. □

Theorem 2.3. *Let γ_a touch AB and AC at F and E, respectively. Let TI meet BC at Z (Figure 7). Then*

$$\frac{BF}{CE} = \frac{BZ}{CZ}.$$

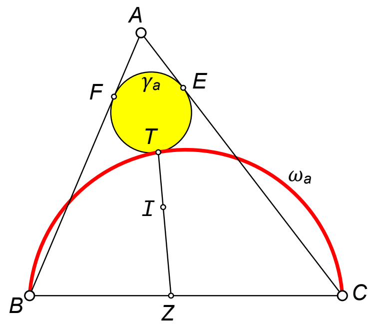


FIGURE 7. $BF/CE = BZ/CZ$

Proof. By Lemma 2.2, $BF/CE = BT/CT$. By Protasov's Theorem, TI bisects $\angle BTC$. Since TZ is an angle bisector of $\triangle BTC$, we have $BT/TC = BZ/CZ$. Hence $BF/CE = BT/CT = BZ/CZ$. □

The following result comes from [39]. A nice geometric proof can be found in [3]. See also [22].

Theorem 2.4 (The Catalytic Lemma). *Let E be the point where γ_a touches AC. Then E, T, I, and C are concyclic (Figure 8).*

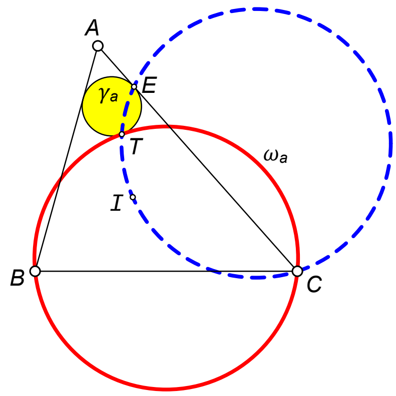


FIGURE 8. E, T, I, C are concyclic

Theorem 2.5. *Let E be the point where γ_a touches AC. Then $\angle BTC = 2\angle IEC$ (Figure 9).*

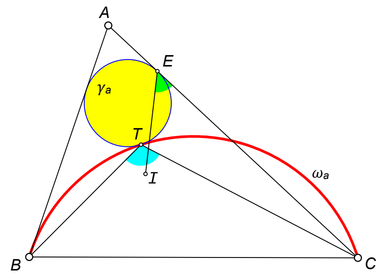


FIGURE 9. blue angle is twice green angle

Proof. By the Catalytic Lemma, E, T, I, and C are concyclic (Figure 10). By Protasov's Theorem, TI bisects $\angle BTC$, so $\angle BTC = 2\angle ITC$. But $\angle ITC = \angle IEC$ because both angles subtend the same arc $\overparen{IC}$. □

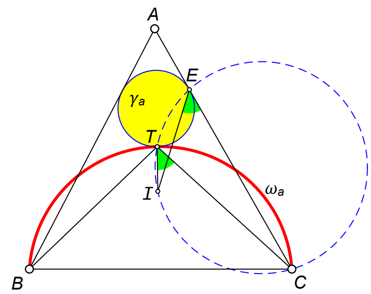


FIGURE 10. green angles are equal

The following result comes from [3].

Theorem 2.6. *Let the touch points of circle γ_a with AC and AB be E and F, respectively. Suppose ω_a meets AC at J between A and C. Let X be the center of the excircle of $\triangle BJC$ opposite C. Then X, F, and E are collinear (Figure 11).*

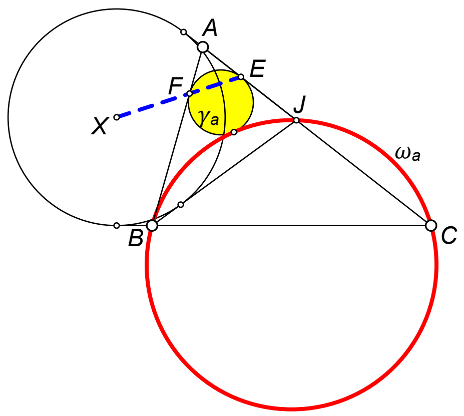


FIGURE 11. X, F, and E are collinear

The following result comes from [32].

Theorem 2.7. *The perpendicular bisector of BC meets ω_a on the opposite side from T at N as shown in Figure 12. Then T, I, and N are collinear.*

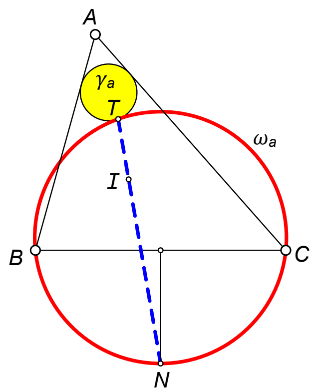


Figure 12.

Proof. From Protasov's Theorem, TI bisects $\angle BTC$. Thus, TI intersects the arc $\widehat{BC}$ (not containing T) at its midpoint. This midpoint lies on the perpendicular bisector of BC and we are done. □

Note. This theorem provides a nice method for constructing γ_a. First construct N as the intersection of the perpendicular bisector of BC with ω_a. Then construct T as the intersection of NI with ω_a. Finally, the center of γ_a is found as the intersection of the line joining the center of ω_a and T with AI.

Corollary 2.8. *We have $\angle NBC = \angle BCN$.*

The following result is suggested by [17].

Theorem 2.9. *Let N be the midpoint of arc $\widehat{BC}$ opposite T. Let E be the point where γ_a touches side AC (Figure 13). Let J be the point where ω_a meets AC. Then $IE \parallel NJ$.*

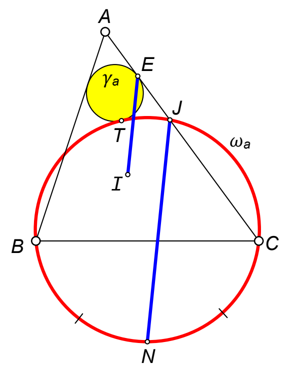


Figure 13. blue lines are parallel

Proof. By Theorem 2.5, $\angle IEC$ is half of $\angle BTC$. But half of $\angle BTC$ is equal to $\angle NTC$ and $\angle NTC = \angle NJC$ because both angles are inscribed in arc $\widehat{NC}$. Thus, $\angle IEC = \angle NJC$ which makes $IE \parallel NJ$. □

The following result comes from [31].

Theorem 2.10. *Let ω_a meet AC at J and let the line through I parallel to BJ meet AC at F. Let E be the point where γ_a touches AC. Then $IF = FE$ (Figure 14).*

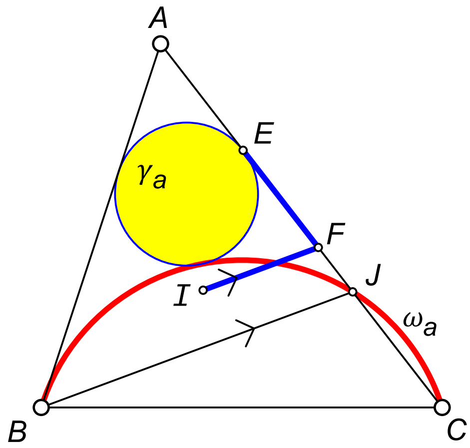


FIGURE 14. blue segments are congruent

Proof. Let TI meet ω_a again at N. By Theorem 2.9, $NJ \parallel IE$ (Figure 15). Thus

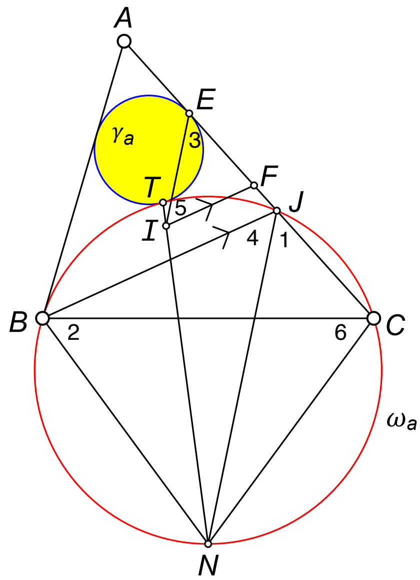


FIGURE 15.

$\angle 3 = \angle 1$. But $\angle 1 = \angle 2$ since both subtend arc $\overset{\frown}{NC}$ in circle ω_a. Hence,

$$\angle 3 = \angle 2. \tag{1}$$

By Corollary 2.8, we have $\angle 2 = \angle 6$. But $\angle 6 = \angle 4$ since both subtend arc $\overset{\frown}{BN}$. Thus,

$$\angle 2 = \angle 4. \tag{2}$$

Since $IE \parallel NJ$ and $IF \parallel BJ$, we can conclude that

$$\angle 4 = \angle 5. \tag{3}$$

Combining equations (1), (2), and (3), we find that

$$\angle 3 = \angle 2 = \angle 4 = \angle 5,$$

so $\angle 3 = \angle 5$. Thus, $\triangle FIE$ is isosceles with $IF = FE$. □

For other proofs, see [4] and [5].

This theorem provides another simple way to construct circle γ_a. Draw the line through I parallel to BJ to get point F where this line meets AC. With center F, draw a circle with radius FI. Let this circle meet AC (nearer A) at point E. This is the touch point for circle γ_a. Erect a perpendicular at E to AC. This perpendicular meets AI at the center of γ_a.

Theorem 2.11. *Let T be any point on arc $\widehat{BC}$. Let F be the foot of the perpendicular from T to BC (Figure 16). Then $\angle BTF = \angle O_aTC$.*

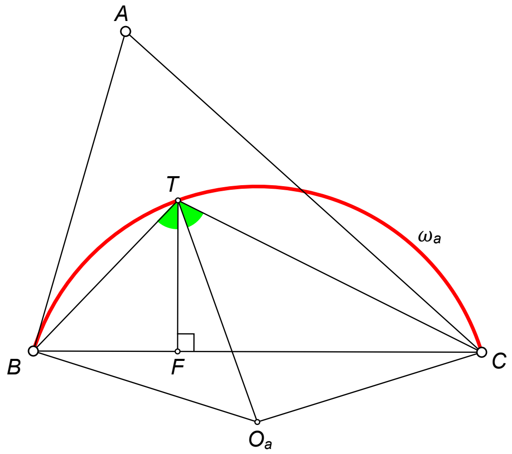
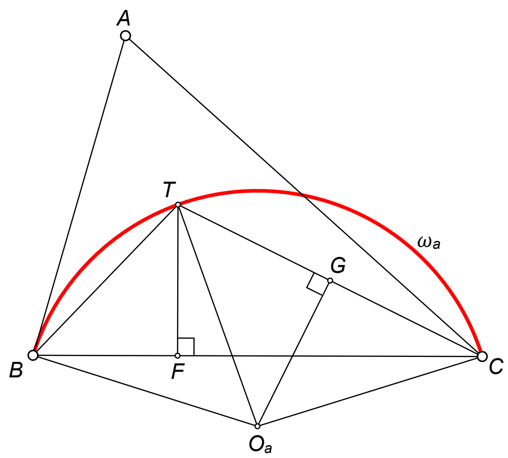


Figure 16. green angles are equal

Proof. Let G be the foot of the perpendicular from O_a to TC. Since $\angle CBT$ is measured by half the measure of $\widehat{TC}$ and $\angle TO_aC$ equals the measure of $\widehat{TC}$, we have

$$\angle FBT = \frac{1}{2}\angle CO_aT = \angle GO_aT.$$

Complements of equal angles are equal, so $\angle BTF = \angle O_aTC$. □



Theorem 2.12. *Let M be the midpoint of BC (Figure 17). Then* $\angle MO_aT = 2\angle ITO_a$.

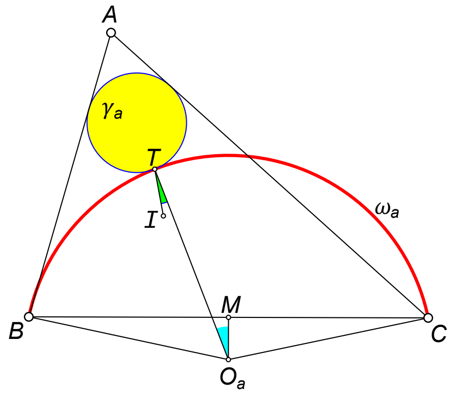
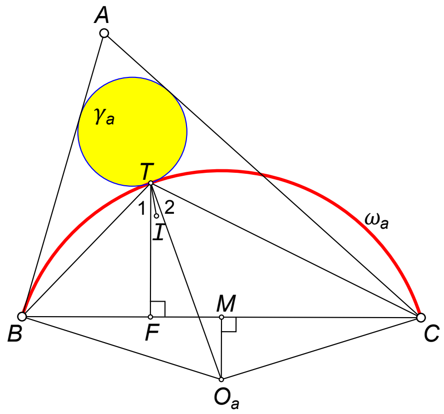


FIGURE 17. blue angle = twice green angle



Proof. Let F be the foot of the perpendicular from T to BC. By Theorem 2.11, $\angle 1 = \angle 2$ in the figure to the right.

By Protasov's Theorem, $\angle BTI = \angle ITC$. Therefore $\angle FTI = \angle ITO_a$ or

$$\angle ITO_a = \frac{1}{2}\angle FTO_a = \frac{1}{2}\angle MO_aT$$

since $TF \parallel MO_a$.

Hence, $\angle MO_aT = 2\angle ITO_a$. □

Theorem 2.13. *Let IT meet γ_a again at T' (Figure 18). Then $T'D \perp BC$.*

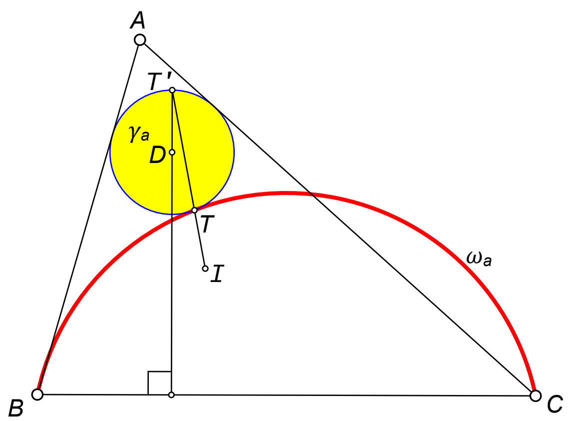


FIGURE 18. $T'D \perp BC$

The following proof is due to Biro Istvan.

Proof. Since γ_a and ω_a are tangent at T, this means D, T, and O_a are collinear. Since I, T, and T' are also collinear, we find $\angle T'TD = \angle ITO_a$. Extend TI until it meets ω_a again at N (Figure 19).

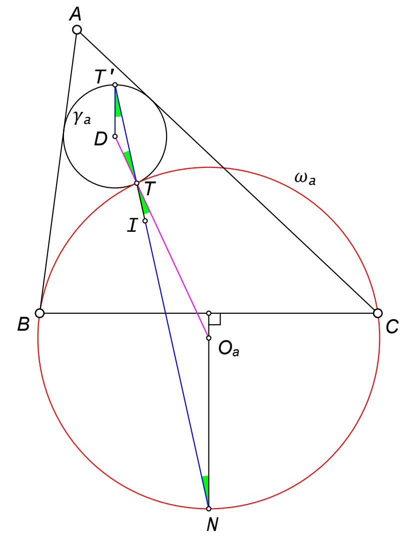


FIGURE 19.

By Theorem 2.7, $NO_a \perp BC$. Base angles of an isosceles triangle are equal and vertical angles are equal, so $\angle DT'T = \angle T'TD = \angle ITO_a = \angle O_aNT$. So $T'D \parallel O_aN$ because $\angle DT'N = \angle O_aNT$. Thus, $T'D \perp BC$. $\square$

Theorem 2.14. *Let IT meet γ_a again at T'. Let E be the point where γ_a touches side AC (Figure 20). Then $\angle EDT' = \angle ACB$.*

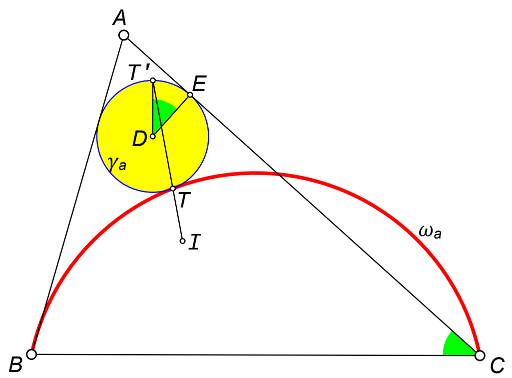
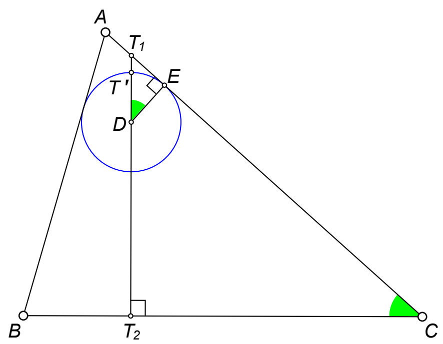


FIGURE 20. green angles are equal

Proof. Let DT' meet AC at T_1 and let DT' meet BC at T_2. By Theorem 2.13, $T_1T_2 \perp BC$. From right triangles T_1ED and CT_2T_1, we see that $\angle EDT' = \angle ACB$ since they are both complementary to $\angle T_2T_1C$. $\square$



3. Properties Related to the Incircle

In this section, we will discuss properties of Ajima circles that are related to the incircle. As before, I will denote the incenter of $\triangle ABC$. Obviously, $IL \perp BC$.

Throughout this section, points will be labeled as shown in Figure 21 and described in the following table.

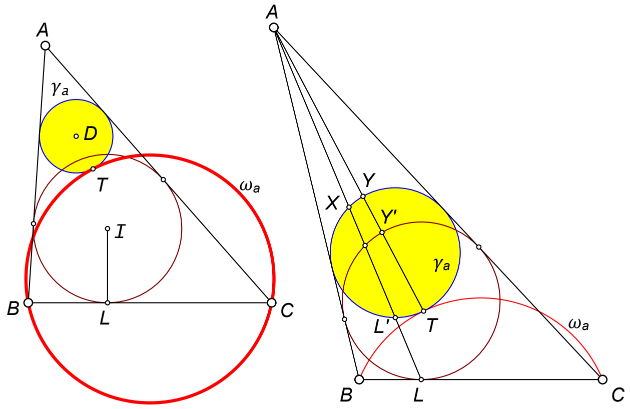


FIGURE 21. basic configuration plus incircle

Notation for this Section	
Symbol	**Description**
I	incenter of $\triangle ABC$
D	center of γ_a
T	Unless specified otherwise, T is the point where γ_a touches ω_a.
L	point where incircle touches BC
L'	point closer to L where AL meets γ_a
X	point closer to A where AL meets γ_a
Y	point where AT meets γ_a again
Y'	point where AT meets the incircle

Theorem 3.1. *Let γ_a be any circle inscribed in $\angle BAC$. Let AL meet γ_a at L' (closer to L). Let D be the center of γ_a. Then $DL' \perp BC$ (Figure 22).*

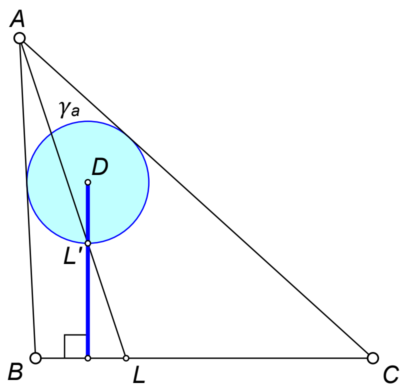
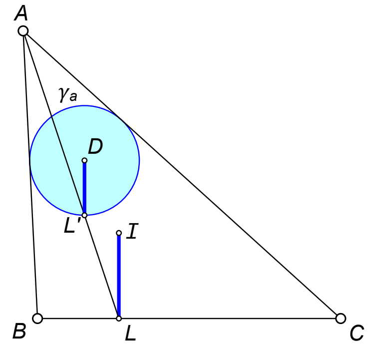


FIGURE 22. $DL' \perp BC$

Proof. The incircle and circle γ_a are homothetic with A being the center of the homothety. This homothety maps D to I and maps L' to L. Since a homothety maps lines into parallel lines, we can conclude that $DL' \parallel IL$. Since $IL \perp BC$, we therefore have $DL' \perp BC$. □



Theorem 3.2. *Let γ_a be any circle inscribed in $\angle BAC$. Let AL meet γ_a at L' (closer to L). Then the tangent to γ_a at L' is parallel to BC (Figure 23).*

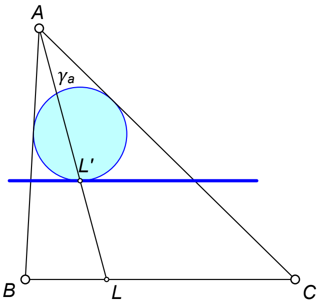


Figure 23. blue tangent is parallel to BC

Proof. The tangent at L' is perpendicular to DL' (Figure 22) which is also perpendicular to BC by Theorem 3.1. □

Theorem 3.3. *Let γ_a be any circle inscribed in $\angle BAC$. Let D be the center of γ_a. Let T be any point on γ_a. Let AL meet γ_a at L' (closer to L). Let AT meet γ_a at Y and Y' with Y closer to A. Let AT extended meet the incircle again at T' (Figure 24). Then $YL' \parallel Y'L$, $L'T \parallel LT'$, and $YD \parallel Y'I$.*

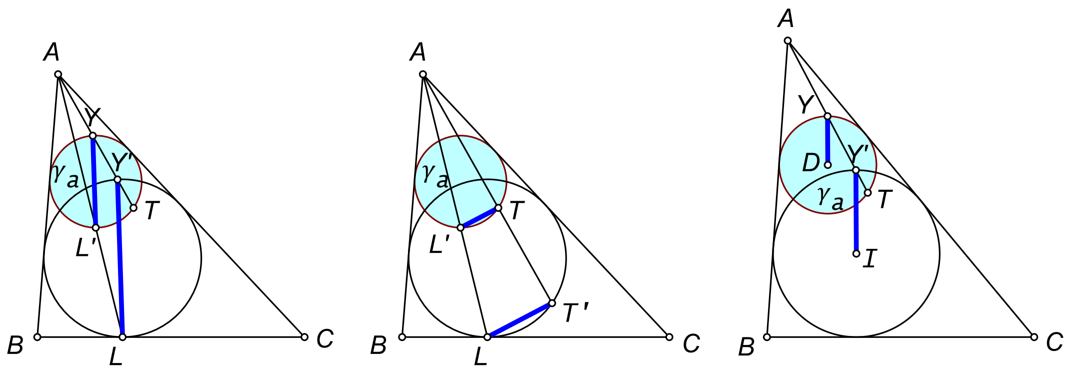


Figure 24. blue lines are parallel

Proof. The incircle and circle γ_a are homothetic with A being the center of the homothety. This homothety maps D to I, L' to L, Y to Y', and T to T'. These results then follow because a homothety maps a line into a parallel line. □

Theorem 3.4. *We have $\angle XTL' = \angle XLB$ (Figure 25).*

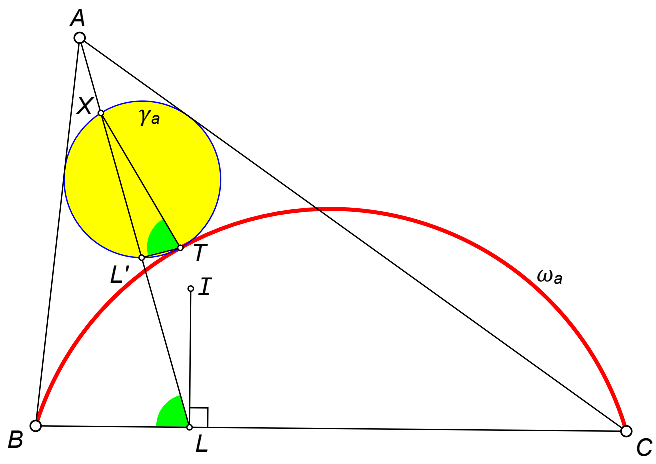


FIGURE 25. green angles are equal

Proof. This is a special case of the following more general theorem. □

Theorem 3.5. *Let T be any point on γ_a, on the opposite side of AL from B. Let AL meet γ_a at X and L' (with X nearer A). Then $\angle XTL' = \angle XLB$.*

Proof. Let $L'Z$ be the tangent to γ_a at L' as shown in Figure 26.

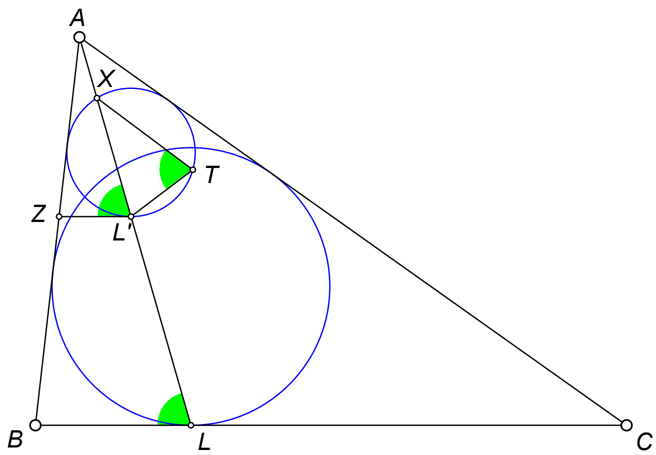


FIGURE 26. green angles are equal

From Theorem 3.2, $L'Z \parallel LB$, so $\angle AL'Z = \angle ALB$. But $\angle XTL' = \angle XL'Z$ since both are measured by half of arc $\widehat{XL'}$. Thus $\angle XTL' = \angle ALB = \angle XLB$. □

Theorem 3.6. *Let γ_a be any circle inscribed in $\angle BAC$. Let T be any point on γ_a, on the opposite side of AL from B. Let AL meet γ_a at X and L' (with X nearer A) as shown in Figure 27. Let TL' meet CB at K. Then X, T, L, and K are concyclic.*

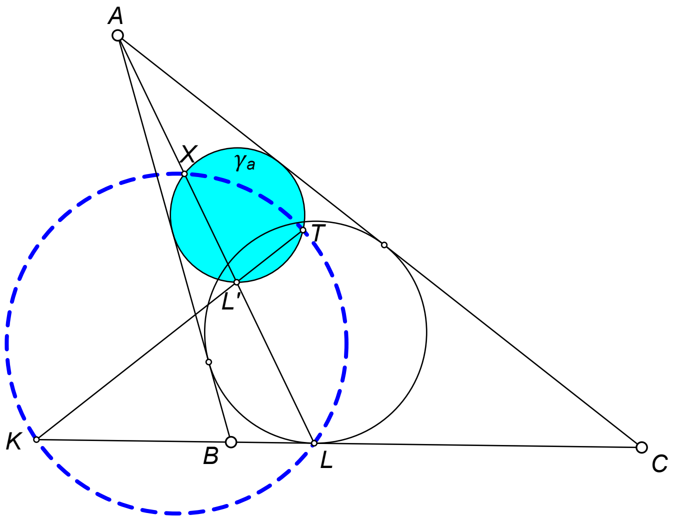


FIGURE 27. X, T, L, K lie on a circle.

Proof. From Theorem 3.5, $\angle XTL' = \angle XLB$, or equivalently, $\angle XTK = \angle XLK$. Thus, X, T, L, and K are concyclic. $\square$

The next five results have been suggested by Navid Safaei.

Lemma 3.7. *Let N be the midpoint of arc $\widehat{BC}$ of a circle. Let T be a point on arc $\widehat{BN}$. Then TN is the external angle bisector of $\angle BTC$ (Figure 28).*

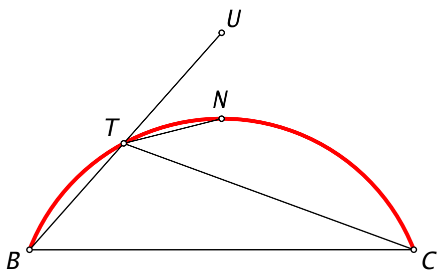


FIGURE 28.

Proof. Using properties of angles inscribed in a circle, we have
$$\angle NTU = \frac{1}{2}(\widehat{BT} + \widehat{TN}) = \frac{1}{2}\widehat{BN} = \frac{1}{2}\widehat{CN} = \angle CTN,$$
so TN bisects $\angle CTU$. $\square$

Lemma 3.8. *Let N be the midpoint of arc $\widehat{BC}$ of ω_a. Then L', T, and N are collinear (Figure 29).*

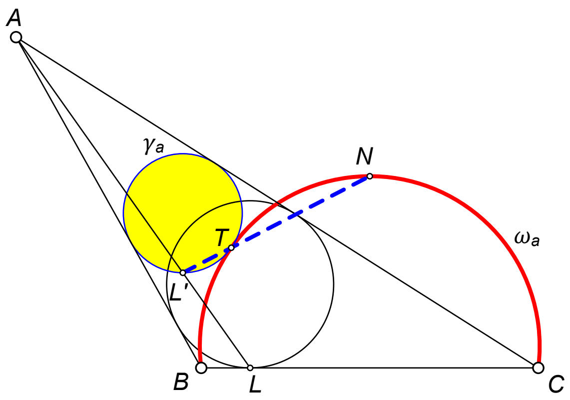


FIGURE 29.

Proof. Note that T is the center of a homothety between γ_a and ω_a. Since the tangents at L' and N are parallel to BC (by Theorem 3.2), this means that they are corresponding points of the homothety and hence $L'N$ passes through the center of the homothety, T. □

Theorem 3.9. *The line TL' is the exterior angle bisector of $\angle BTC$ in $\triangle BTC$. (Figure 30).*

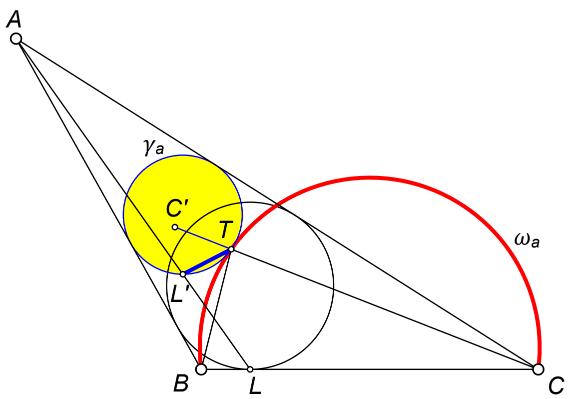


FIGURE 30. TL' bisects $\angle C'TB$

Proof. This follows immediately from Lemmas 3.7 and 3.8. □

Theorem 3.10. *Let E and F be the points where γ_a touches AC and AB, respectively. Then EF, TL', and CB are concurrent.*

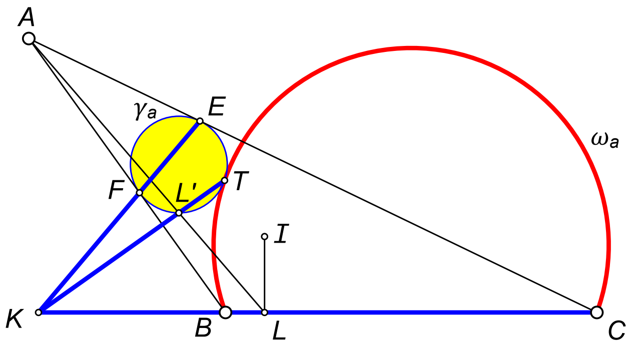


FIGURE 31. Blue lines are concurrent.

Proof. Let EF meet CB at K. From Theorem 2.3, we have

$$\frac{BT}{CT} = \frac{BF}{CE}. \tag{4}$$

By Menelaus' Theorem applied to $\triangle ABC$ and the transversal FE, we have

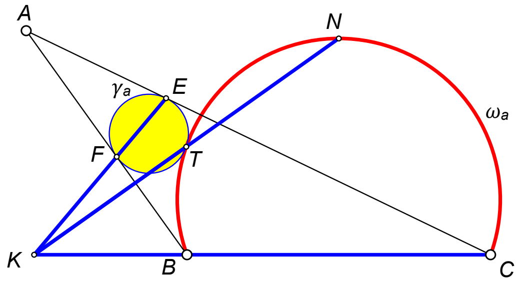
$$\frac{BF}{FA} \cdot \frac{AE}{EC} \cdot \frac{CK}{KB} = -1$$

from which we get

$$\frac{BF}{CE} = \frac{KB}{CK} \tag{5}$$

because $FA = AE$. From equations (4) and (5) we get

$$\frac{KB}{CK} = \frac{BF}{CE} = \frac{BT}{CT}. \tag{6}$$



Hence (by a property of external angle bisectors), TK is the external angle bisector of $\angle BTC$ in $\triangle BTC$. Let N be the midpoint of arc $\widehat{BC}$ as shown in the figure above. By Lemma 3.7, TN is also the external angle bisector of $\angle BTC$. It follows that N, T, and K are collinear. Since TK passes through both L' and N (by Lemma 3.8), the four points, K, L', T, and N lie on a line, so EF, TL', and CB all pass through K. □

Theorem 3.11. *The lines YX, TL', and CB are concurrent (Figure 32).*

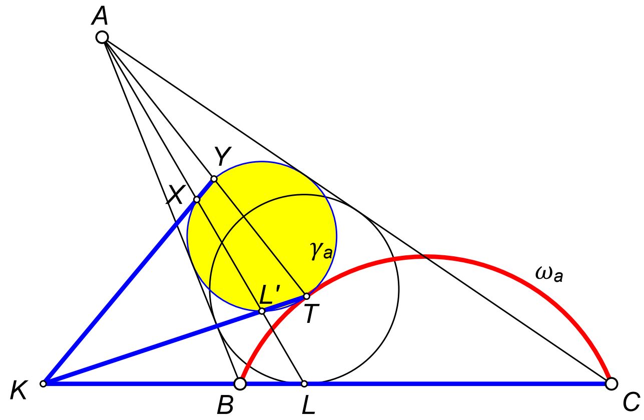


FIGURE 32. Blue lines are concurrent.

Proof. Let E and F be the points where γ_a touches AC and AB, respectively. Let EF meet CB at K. Then EF, TL', and CB are concurrent at K by Theorem 3.10 (Figure 33).

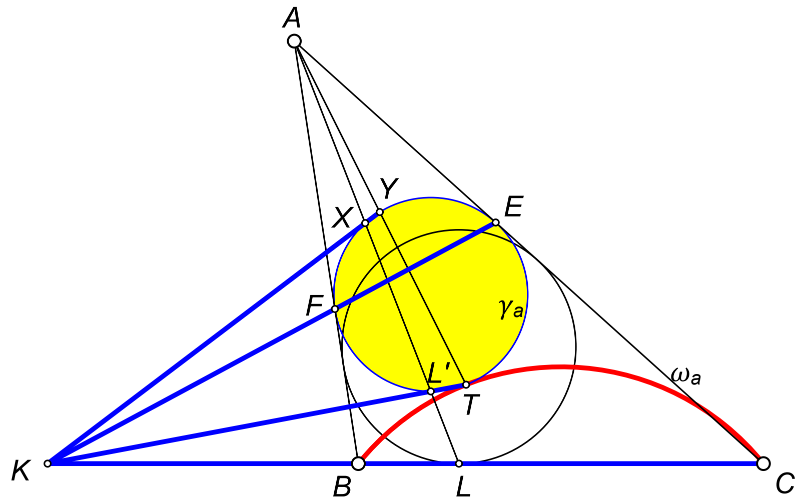


FIGURE 33. Blue lines are concurrent.

Since AF and AE are tangents to circle γ_a, this means that EF is the polar of A with respect to γ_a. Since AXL' and AYT are two secants from A, this means that YX meets TL' on the polar of A (line EF). But K is the only point on TL' that lies on the polar of A. Thus, YX also passes through K. □

The following result comes from [28].

Theorem 3.12. *The line TL' bisects $\angle ATL$ (Figure 34).*

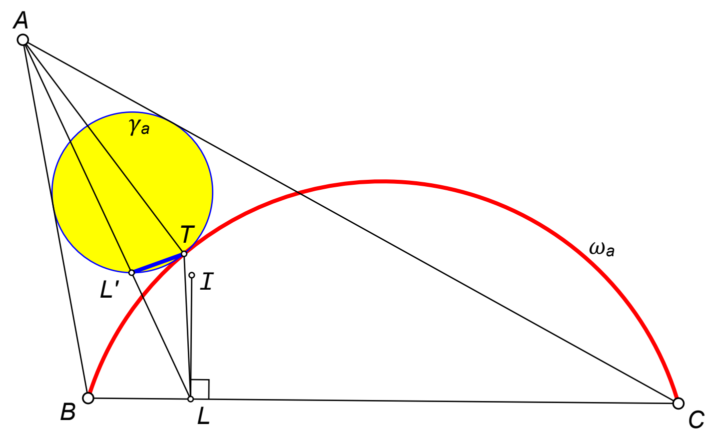


Figure 34. TL' bisects $\angle ATL$

Proof. Let TL' meet CB at K (Figure 35).

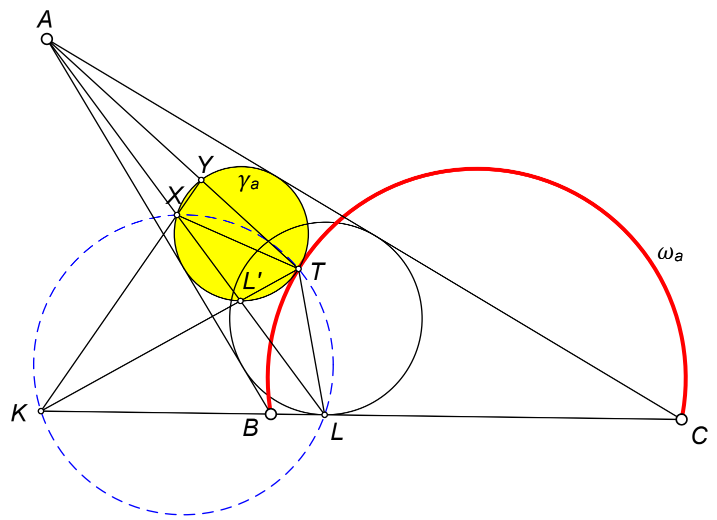


Figure 35.

By Theorem 3.11, YX passes through K. By Theorem 3.6, $XTLK$ is a cyclic quadrilateral, so $\angle KTL = \angle KXL$. But since $XYTL'$ is also a cyclic quadrilateral, $\angle KXL = \angle YTL'$. Thus, $\angle KTL = \angle YTL'$ so TL' bisects $\angle ATL$. $\square$

Theorem 3.13. *Extend AT until it meets the incircle at T' as shown in Figure 36. Then $TL = TT'$.*

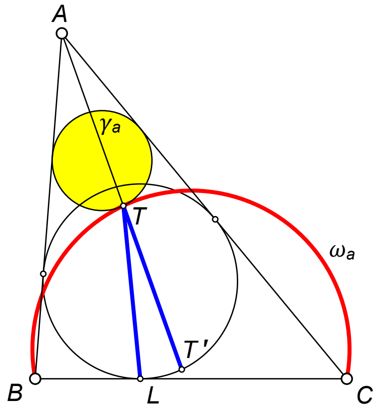
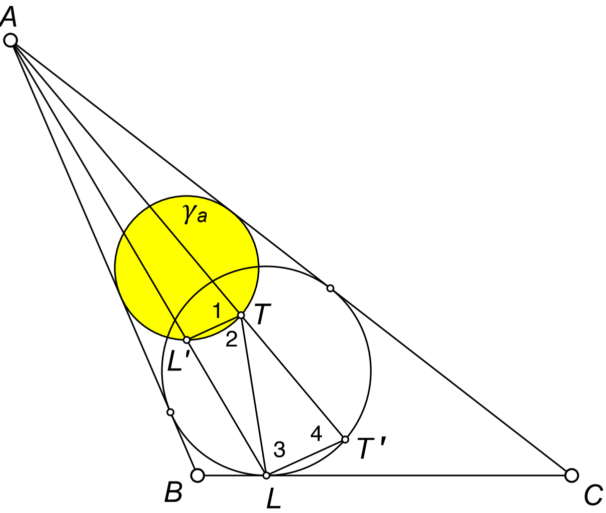


FIGURE 36. blue lines are congruent



Proof. Let AL meet γ_a at L', closer to L, as shown in the figure to the right. The incircle and γ_a are homothetic with A being the center of the homothety. Since a homothety maps a line into a parallel line, $L'T \parallel LT'$. Thus $\angle 2 = \angle 3$ and $\angle 1 = \angle 4$. By Theorem 3.12, $\angle 1 = \angle 2$. Hence $\angle 3 = \angle 4$ making $\triangle TLT'$ an isosceles triangle. $\square$

The following result comes from [33].

Theorem 3.14. *Let AT meet BC at M. Then TI bisects $\angle LTM$ (Figure 37).*

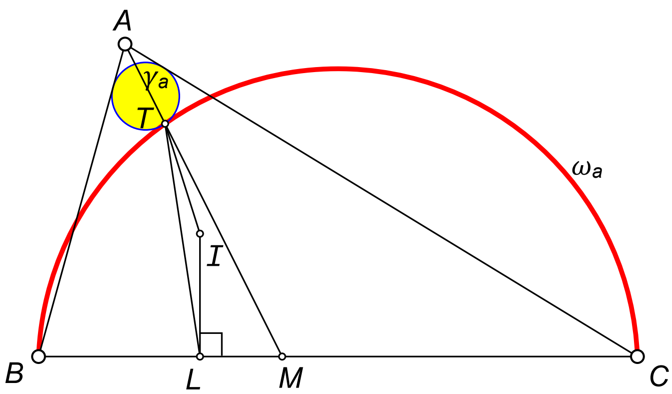
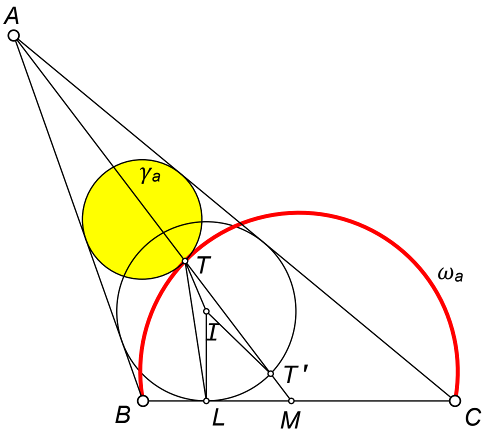


FIGURE 37. TI bisects $\angle LTM$

Proof. Let AM meet the incircle at T' (closer to M) as shown in the figure to the right. By Theorem 3.13, $TL = TT'$. Since both L and T' lie on the incircle, we must have $IL = IT'$. Thus $\triangle LTI \cong \triangle T'TI$ by SSS. Hence $\angle LTI = \angle ITT'$. $\square$



Theorem 3.15. *We have $\angle ATD = \angle ILT$ (Figure 38).*

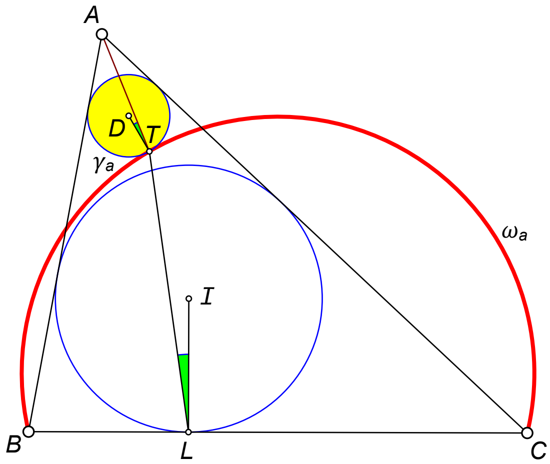
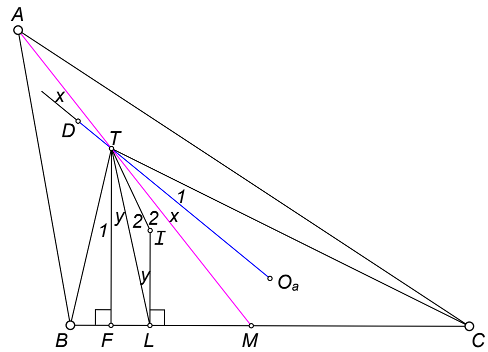


FIGURE 38. green angles are equal

Proof. Let F be the foot of the perpendicular from T to BC. Let AT meet BC at M. Since γ_a is tangent to ω_a at T, this means that DTO_a is a straight line.

Number the resulting angles as shown in the figure to the right. Lines AM and DO_a meet at T forming equal vertical angles. These are labeled x in the figure. From Theorem 2.11, $\angle BTF = \angle O_aTC$. These are labeled 1 in the figure. Since $TF \parallel IL$, $\angle FTL = \angle ILT$. These are labeled y in the figure. From Theorem 3.14, $\angle LTI = \angle ITM$. These are labeled 2 in the figure.



By Protasov's Theorem, $1+y+2 = 2+x+1$. Thus $x = y$ and $\angle ATD = \angle ILT$. $\square$

Lemma 3.16. *Two circles, C_1 and C_2, are internally tangent at P. A chord AB of C_1 meets C_2 at points C and D as shown in Figure 39. Then $\angle APC = \angle DPB$.*

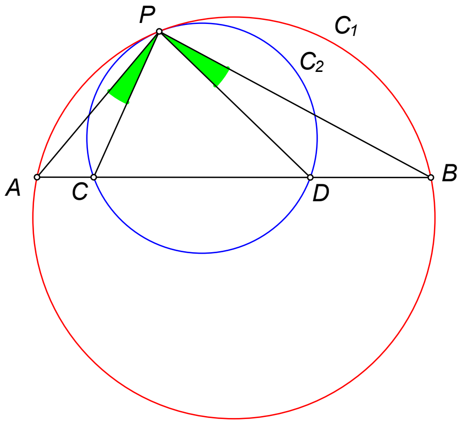
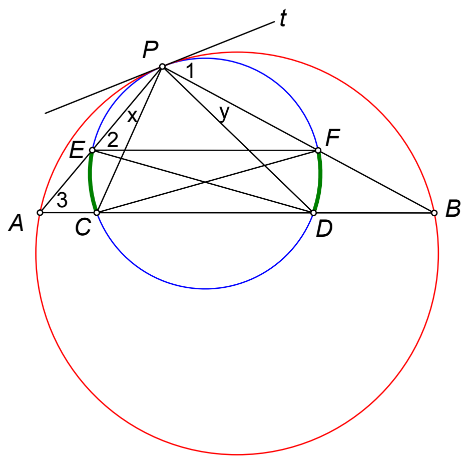


FIGURE 39. green angles are equal



Proof. Let t be the common tangent at P. Let PA meet C_2 at E and let PB meet C_2 at F. Label the angles as shown in the figure to the right.

In the blue circle, $\angle 1 = \angle 2$ since both are measured by half of arc $\widehat{PF}$. In the red circle, $\angle 1 = \angle 3$ since both are measured by half of arc $\widehat{PB}$.

Thus $\angle 2 = \angle 3$ which makes $EF \parallel AB$. Parallel chords intercept equal arcs, so $\widehat{CE} = \widehat{FD}$ which implies $\angle x = \angle y$. □

The following result comes from [9].

Theorem 3.17. *Let AT meet BC at M. The $\odot TLM$ is tangent to γ_a (Figure 40).*

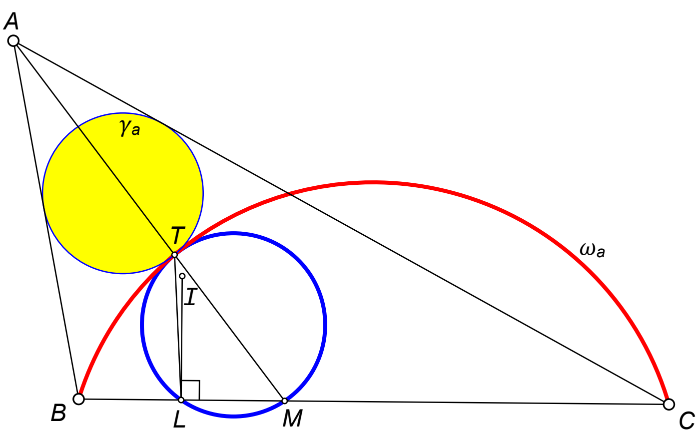


FIGURE 40. three circles touch at T

Proof. Let Γ be the circle tangent to ω_a at T and passing through L. Let LC meet Γ again at M' as shown in Figure 41. By Lemma 3.16,

$$\angle BTL = \angle M'TC.$$

These are labeled "1" in the figure. By Protasov's Theorem,

$$\angle BTI = \angle ITC.$$

Subtracting shows that

$$\angle 2 = \angle 3.$$

So TI bisects $\angle LTM'$. But by Theorem 3.14, TI bisects $\angle LTM$. This implies that $M' = M$, so $\Gamma = \odot(TLM)$ and we are done. □

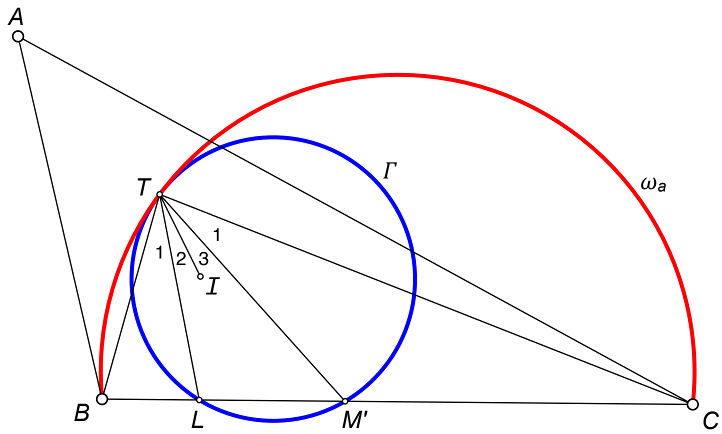


FIGURE 41.

Theorem 3.18. *We have $\angle DTI = \angle TIL$ (Figure 42).*

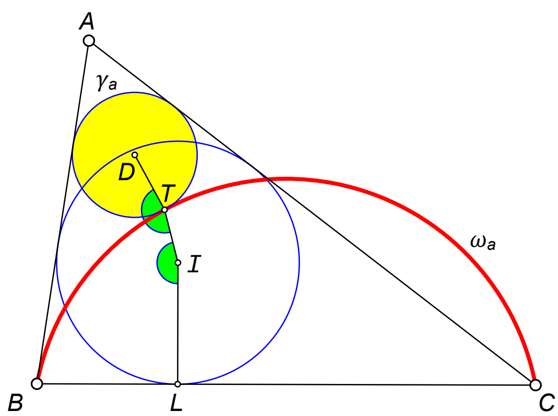


FIGURE 42. green angles are equal

Proof. Let AT meet BC at M. The sum of the angles of $\triangle TIL$ is $180°$, so

(7) $\quad 180° - \angle LTI = \angle TIL + \angle TLI.$

Since ATM is a straight line, we have

$$\angle DTI = 180° - \angle ITM - \angle ATD.$$

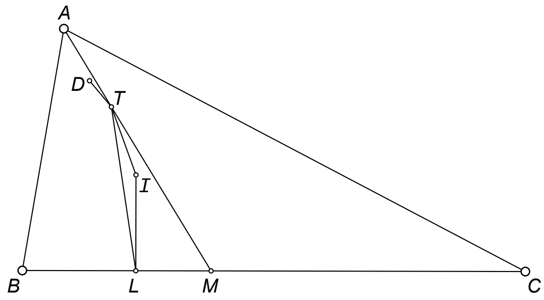
By Theorem 3.14, $\angle ITM = \angle LTI$, so

$$\angle DTI = 180° - \angle LTI - \angle ATD.$$

From equation (7), we get

$$\angle DTI = \angle TIL + \angle TLI - \angle ATD.$$

From Theorem 3.15, $\angle TLI = \angle ATD$. Hence $\angle DTI = \angle TIL$. ☐



Lemma 3.19. *Let γ_a be any circle inscribed in $\angle BAC$. Let T be any point on γ_a on the other side of AL from B. Let AL meet γ_a at X (nearer A). Let AT meet the incircle at Y'. Then X, Y', T, and L are concyclic (Figure 43).*

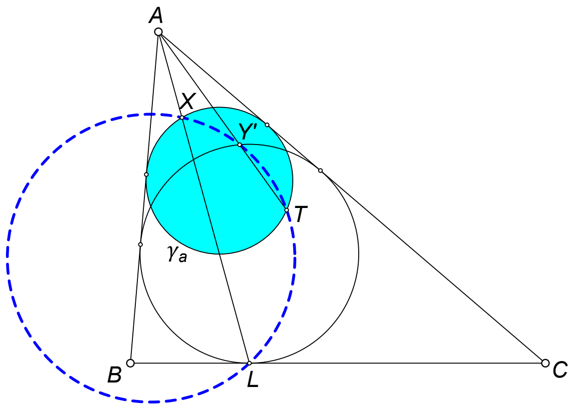


FIGURE 43. four points lie on a circle

Proof. Let L' be the point (nearer L) where AL meets γ_a as shown in the figure to the right. Lines AXL' and AYT are both secants to γ_a, so

$$AX \cdot AL' = AY \cdot AT.$$

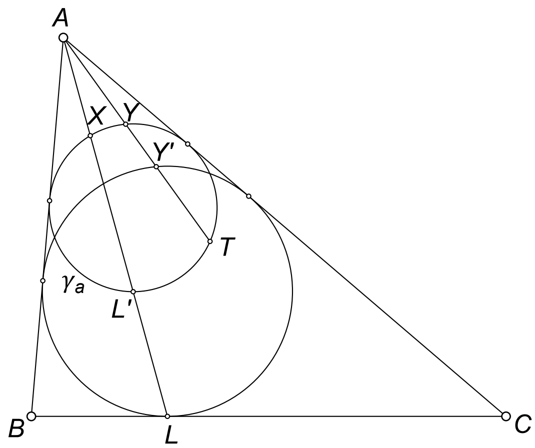
Note that the incircle and circle γ_a are homothetic with A as the center of the homothety. This homothety maps L' to L and maps Y to Y'. Therefore,

$$\frac{AL'}{AL} = \frac{AY}{AY'}.$$

Hence

$$\frac{AX}{AT} = \frac{AY}{AL'} = \frac{AY'}{AL}.$$

Thus $AX \cdot AL = AY' \cdot AT$ which implies that X, Y', T, and L lie on a circle. ☐



The following result comes from [35].

Theorem 3.20. *We have $IT \perp TL'$ (Figure 44).*

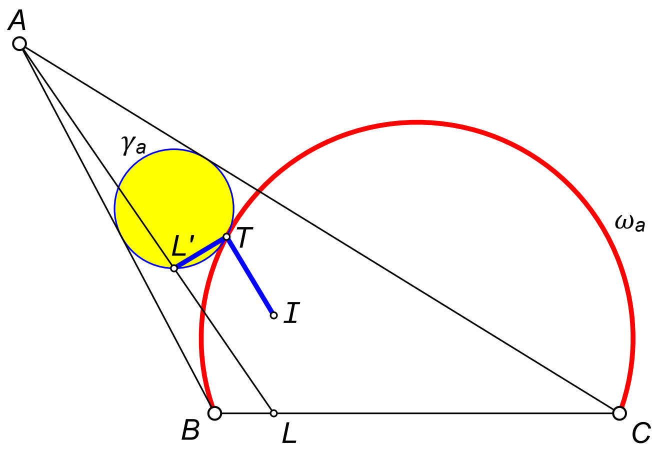


Figure 44. blue lines are perpendicular

Proof. Line TL' is the external angle bisector of $\angle BTC$ by Theorem 3.9. Line TI is the internal angle bisector of $\angle BTC$ by Protasov's Theorem. Thus, $IT \perp TL'$. ☐

Theorem 3.21. *Let γ_a be any circle inscribed in $\angle BAC$. Let AL meet γ_a at X (closer to A). Let γ_a touch AC and AB at E and F, respectively. Let AI meet EF at G. Then X, G, I, and L lie on a circle (Figure 45).*

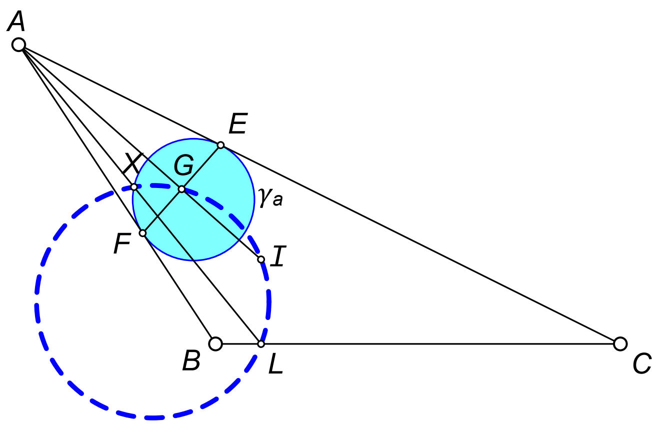


Figure 45. four points lie on a circle

Proof. Let XL meet γ_a again at L'. Line AXL' is a secant to circle γ_a, and AE a tangent. So $AX \cdot AL' = (AE)^2$.

Triangles AGE and AED are similar right triangles, so

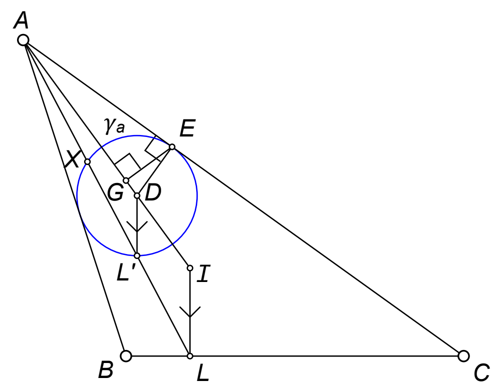
$$\frac{AE}{AG} = \frac{AD}{AE} \quad \text{or} \quad (AE)^2 = AD \cdot AG.$$

Thus,

$$AX \cdot AL' = AD \cdot AG \quad \text{or} \quad \frac{AX}{AG} = \frac{AD}{AL'}.$$

From Theorem 3.1, $DL' \parallel IL$, so

$$\frac{AD}{AL'} = \frac{AI}{AL}.$$



Therefore,

$$\frac{AX}{AG} = \frac{AI}{AL} \quad \text{or} \quad AX \cdot AL = AG \cdot AI.$$

This implies that X, G, I, and L lie on a circle. □

Theorem 3.22. *Let γ_a touch AC and AB at E and F, respectively. Let AI meet EF at G. Let EF meet CB at K. Then X, G, Y', T, I, L, and K lie on a circle with diameter KI (Figure 46).*

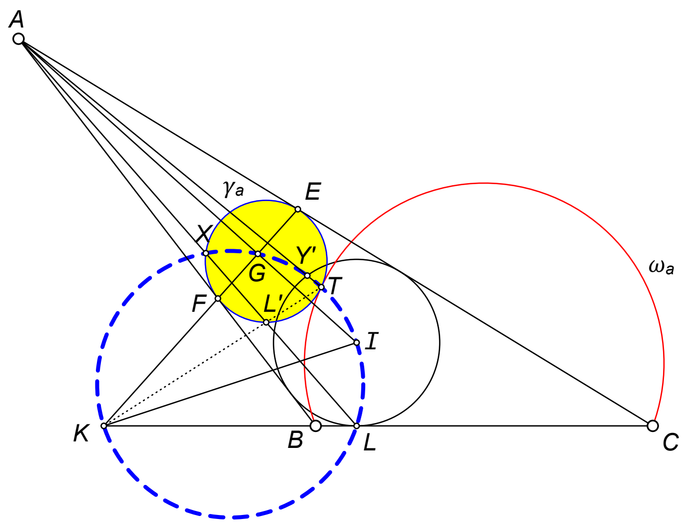


FIGURE 46. seven points lie on a circle

Proof. From Theorem 3.10, TK passes through L'. But from Theorem 3.20, $TL' \perp TI$, so $\angle KTI$ is a right angle. This means that T lies on the circle with diameter KI. Since $\angle ILK$ is also a right angle, this means that L is also on this circle.

From Theorem 3.6, the circle through T, L, and K also passes through X.

From Lemma 3.19, the circle through X, T, and L passes through Y'.

Since AI bisects $\angle BAC$, G is the midpoint of EF and $AG \perp EF$. Hence $\angle KGI$ is a right angle. Thus, G lies on the circle with diameter KI.

Therefore, all seven points lie on the blue circle shown in Figure 46. □

See also [8] for a proof that X, G, T, I, L, and K lie on a circle.

Theorem 3.23. *We have $\angle ATD = \angle IXT$ (Figure 47).*

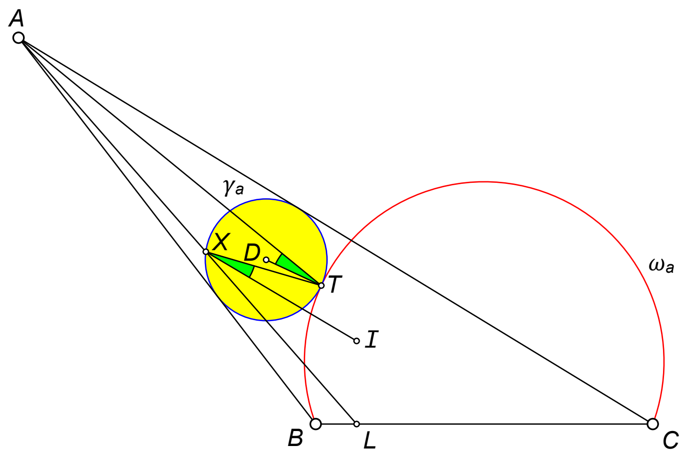
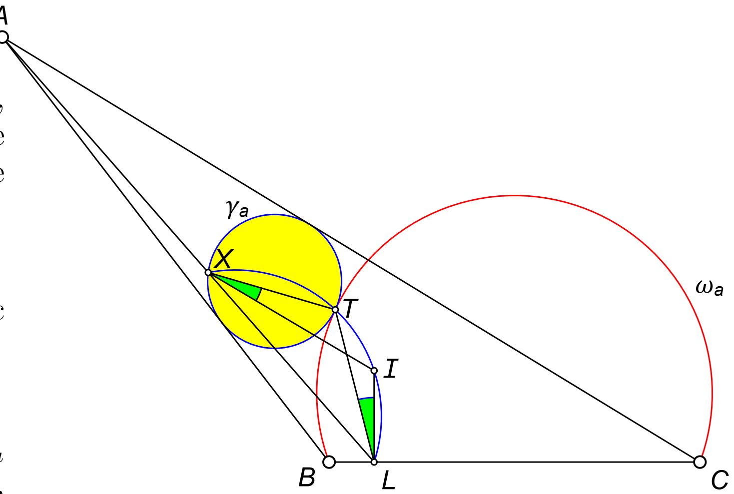


FIGURE 47. green angles are equal



Proof. By Theorem 3.22, X, T, I, and L lie on a circle as shown in the figure to the right. Then

$$\angle IXT = \angle ILT$$

because both subtend arc $\widehat{TI}$. By Theorem 3.15,

$$\angle ILT = \angle ATD$$

where D is the center of γ_a as seen in Figure 47. Thus, $\angle ATD = \angle IXT$. □

4. ARCS WITH A GIVEN ANGULAR MEASURE

We can generalize many of the results in [30] by replacing the semicircles with arcs having the same angular measure. Let ω_a, ω_b, and ω_c be arcs with the same angular measure θ erected internally on the sides of $\triangle ABC$ as shown in Figure 48.

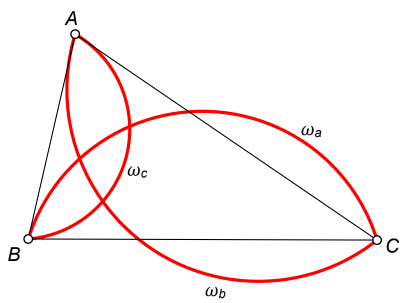


FIGURE 48. arcs have same angular measure

Throughout this section, we will use the symbols shown in the following table.

Notation for this Section	
Symbol	**Description**
θ	angular measure of arc $\widehat{BTC}$
t	$\tan(\theta/4)$
ρ_a	radius of γ_a
ω_a	arc with angular measure θ that passes through B and C
R_a	radius of ω_a
H	point where incircle touches AC
K	point where γ_a touches AC
E	point where AC meets ω_a again
O_a	center of ω_a
L	point where incircle touches BC
L'	point closer to L where AL meets γ_a
X	point closer to A where AL meets γ_a

Theorem 4.1. *Let the line through I parallel to BE meet AC at F (Figure 49). Then*

$$\angle IFA = \frac{\theta}{2}.$$

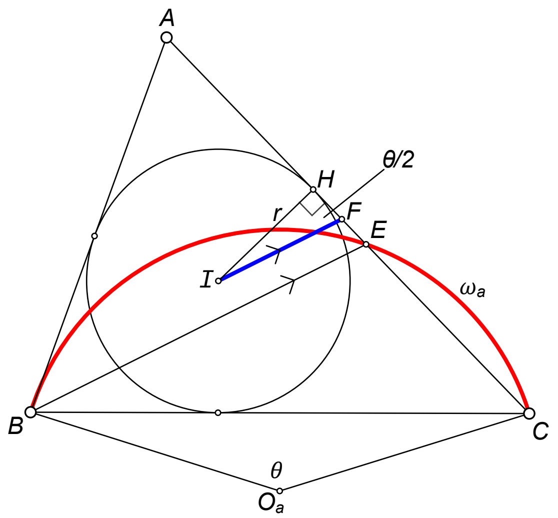


FIGURE 49.

Proof. Since arc $\widehat{BC}$ has measure θ, $\angle CO_aB = \theta$ and the remaining arc on the circle (O_a) outside the triangle must have measure $360° - \theta$. An inscribed angle is measured by half its intercepted arc, so $\angle BEC = 180° - \theta/2$. Consequently, $\angle AEB = \theta/2$ and since $BE \parallel IF$, we have $\angle AFI = \theta/2$. □

Note 1. If $\theta < 2C$, the figure looks different (Figure 50). In this case, the arc (extended) meets AC at a point E such that C lies between A and E. In this case, $\angle CEB$ is measured by half arc $\widehat{BC}$ and $BE \parallel IF$ implies that $\angle AFI = \theta/2$.

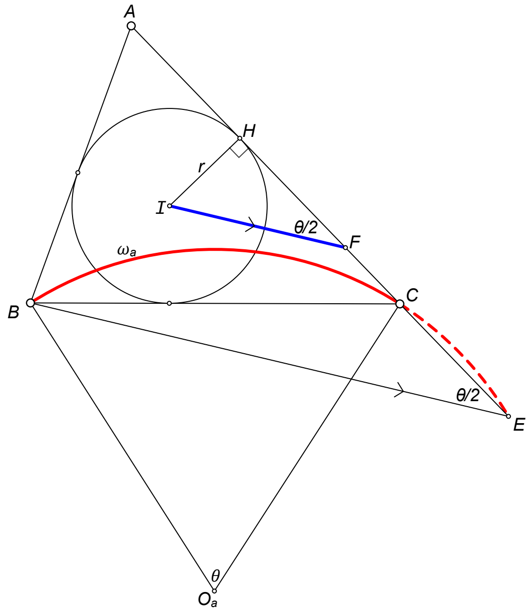


Figure 50. Case $\theta < 2C$

Note 2. If F lies between A and H or if $\theta > 2(180° - A)$ which causes A to lie between E and F, the figure also looks different (Figure 51). In this case, the arc meets AC (possibly extended) at a point E such that F lies between E and H. In this case, the red arc has measure θ and the remaining arc (below BC) has measure $360° - \theta$. Then $\angle BEC$ is measured by half that arc and so $\angle BEC = 180° - \theta/2$. So $BE \parallel IF$ implies that $\angle AFI = \theta/2$.

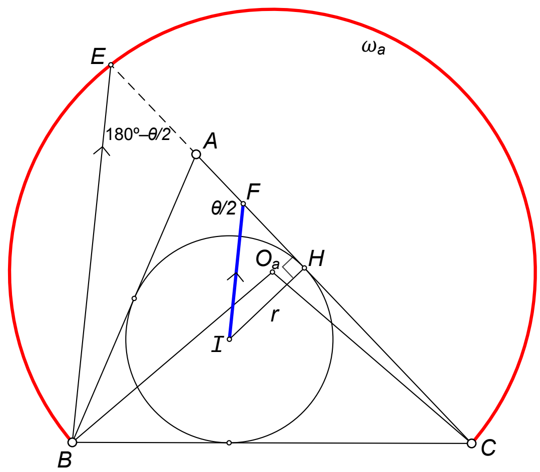


Figure 51. Case $\theta > 2(180° - A)$

Corollary 4.2. *Let the line through I parallel to BE meet AC at F. Then*

$$IF = r \csc \frac{\theta}{2}.$$

Proof. From right triangle IHF, we have

$$\sin\frac{\theta}{2} = \frac{IH}{IF} = \frac{r}{IF},$$

so $IF = r\csc\frac{\theta}{2}$. ◻

Let γ_a be the circle inside $\triangle ABC$ tangent to sides AB and AC and also tangent to ω_a. The radii of circles γ_a, γ_b, and γ_c are denoted by ρ_a, ρ_b, and ρ_c, respectively.

Theorem 4.3. *We have (Figure 52)*

$$\angle HIK = \frac{\theta}{4}.$$

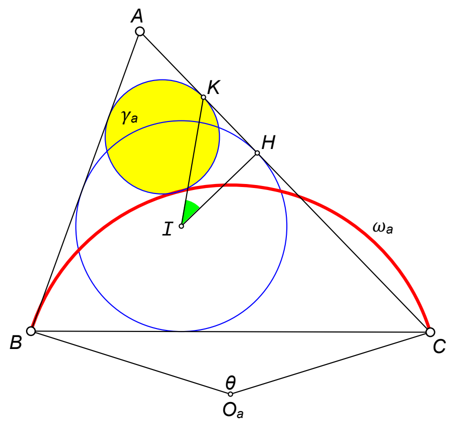


FIGURE 52. green angle $= \theta/4$

Proof. A line through I parallel to BE meets AC at F (Figure 53).

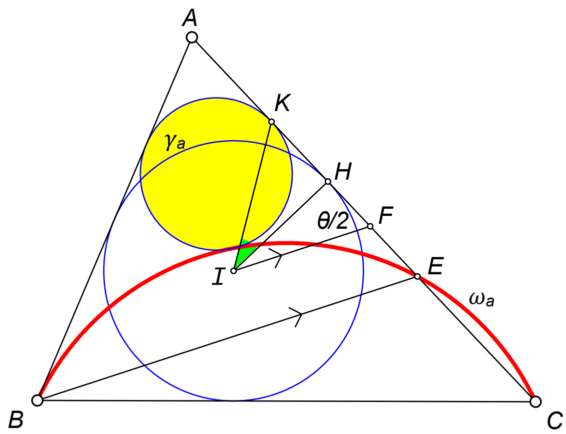


FIGURE 53.

From Theorem 4.1, $\angle AFI = \theta/2$. From Theorem 2.10, $IF = FK$, so $\triangle KFI$ is isosceles and $\angle FIK = \angle IKF = (180° - \theta/2)/2 = 90° - \theta/4$. From right triangle FHI, we see that $\angle FIH = 90° - \theta/2$.

Thus, $\angle HIK = \angle FIK - \angle FIH = (90° - \theta/4) - (90° - \theta/2) = \theta/4$. ◻

Theorem 4.4. *We have $\angle DKI = \theta/4$ (Figure 54).*

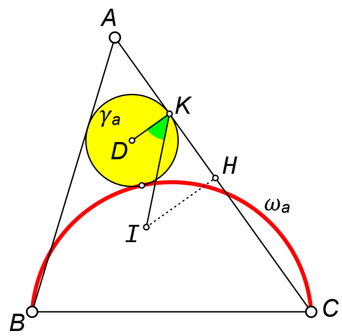


Figure 54. $\angle DKI = \theta/4$

Proof. Since DK and IH are both perpendicular to AC, we have $DK \parallel IH$. Thus $\angle DKI = \angle HIK = \theta/4$. □

Corollary 4.5. *The length of the common external tangent between ρ_a and the incircle is $r\tan\frac{\theta}{4}$.*

Proof. In Figure 52, we see that HK is the common external tangent between ρ_a and the incircle. Since I is the incenter, $IH = r$. From right triangle IHK, we see that

$$\tan\angle HIK = \tan\frac{\theta}{4} = \frac{HK}{IH} = \frac{HK}{r}$$

and the result follows. □

Note. If the arc ω_a gets large enough, point A will lie inside ω_a and circle γ_a will not exist. However, if we expand the definition of γ_a in that case so that it refers to the circle outside $\triangle ABC$, tangent to sides AB and AC extended, and tangent *internally* to ω_a as shown in Figure 55, then Theorem 4.3 still holds.

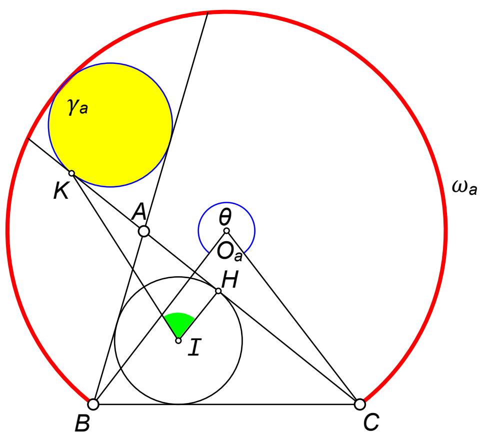


Figure 55. green angle $= \theta/4$

Theorem 4.6 (Ajima's Theorem)**.** *We have*

$$\rho_a = r\left(1 - \tan\frac{A}{2}\tan\frac{\theta}{4}\right). \tag{8}$$

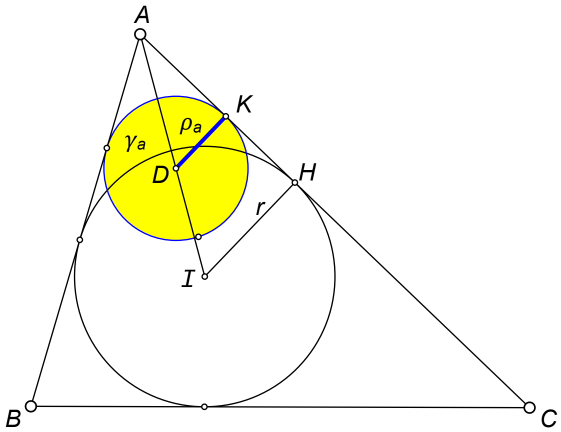


FIGURE 56.

Proof. See Figure 56. By Corollary 4.5,

$$HK = r\tan\frac{\theta}{4}.$$

In right triangle AIH, we have $\angle IAH = A/2$, so

$$AH = r\cot\frac{A}{2}.$$

Therefore,

$$AK = AH - HK = r\cot\frac{A}{2} - r\tan\frac{\theta}{4}.$$

From right triangle AKD with $DK = \rho_a$, we have

$$\rho_a = AK\tan\frac{A}{2} = \left(r\cot\frac{A}{2} - r\tan\frac{\theta}{4}\right)\tan\frac{A}{2}$$

which is the desired result by the identity $\tan x\cot x = 1$. □

The wasan geometer Naonobu Ajima found this result in 1781 (see [13, p. 32]). More info about Ajima's Theorem can be found in [13, pp. 96–97]. It has been said [12, p. 103] that this result is of great importance because it is used in the solution of many Japanese temple geometry problems.

Corollary 4.7. *We have*

$$\rho_a = r\left(1 - \frac{rt}{p-a}\right).$$

Proof. This follows immediately from the well-known fact [37] that in Figure 52, $AH = p - a$, so $\tan(A/2) = r/(p-a)$. □

Corollary 4.8. *We have*

$$\rho_a = \frac{\Delta - (p-b)(p-c)t}{p} = r - \frac{(p-b)(p-c)t}{p}.$$

Proof. We use the well-known formulas $r = \frac{\Delta}{p}$ and $\Delta = \sqrt{(p(p-a)(p-b)(p-c)}$. From Corollary 4.7, we have

$$\begin{aligned}
\rho_a &= r - \frac{r^2 t}{p-a} \\
&= r - \left(\frac{\Delta^2}{p^2}\right)\frac{t}{p-a} \\
&= r - \left(\frac{p(p-a)(p-b)(p-c)}{p^2}\right)\frac{t}{p-a} \\
&= r - \frac{(p-b)(p-c)t}{p} \\
&= \frac{\Delta - (p-b)(p-c)t}{p}.
\end{aligned}$$

□

Theorem 4.6 remains true, with a sign change, if we allow the extended position for γ_a.

Theorem 4.9. *Let ω_a be an arc of a circle with angular measure θ that passes through points B and C of $\triangle ABC$. Suppose $\theta > 2(180° - A)$ so that A lies inside ω_a. Let γ_a be the circle outside the triangle tangent to sides AB and AC extended and also internally tangent to ω_a as shown in Figure 57. Let ρ_a be the radius of γ_a. Then*

$$\rho_a = -r\left(1 - \tan\frac{A}{2}\tan\frac{\theta}{4}\right).$$

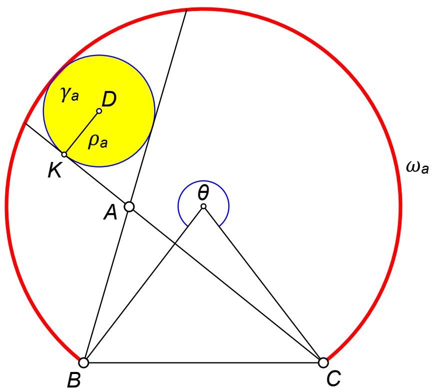


FIGURE 57.

In all cases, we could say that

$$\rho_a = r\left|1 - \tan\frac{A}{2}\tan\frac{\theta}{4}\right|.$$

A unifying discussion about circles tangent to arcs with a given angular measure can be found in [24].

Lemma 4.10. *For any x,*

$$\sin 2x = \frac{2\tan x}{\tan^2 x + 1}.$$

Proof. We have

$$\frac{2\tan x}{\tan^2 x + 1} = \frac{2\tan x}{\sec^2 x} = \frac{2(\sin x)/(\cos x)}{1/\cos^2 x} = 2\sin x\cos x = \sin 2x. \qquad \square$$

Theorem 4.11 (Radius of ω_a). *We have*

$$R_a = \frac{a}{2}\csc\frac{\theta}{2}.$$

Proof. Let M be the foot of the perpendicular from O_a to BC (Figure 58).

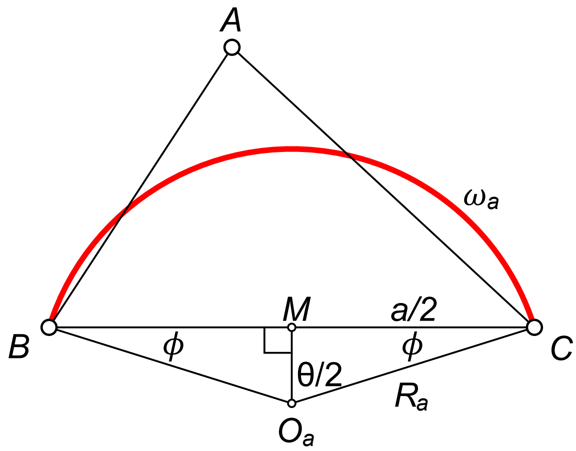


FIGURE 58.

Then $O_aC = R_a$ and $MC = a/2$. We have $\angle CO_aB = \theta$ since the angular measure of the arc is θ. Thus $\angle CO_aM = \theta/2$ and hence $\sin(\theta/2) = (a/2)/R_a$ and the result follows. $\square$

Corollary 4.12. *We have*

$$R_a = \frac{a(t^2+1)}{4t}. \tag{9}$$

Proof. This follows from Lemma 4.10. $\square$

Corollary 4.13. *We have*

$$R_a = \frac{R(t^2+1)\sin A}{2t}. \tag{10}$$

Proof. From the Extended Law of Sines, we have $a/\sin A = 2R$. Substituting $a = 2R\sin A$ into equation (9) gives the desired result. $\square$

Theorem 4.14. *We have (Figure 59)*

$$\frac{AL'}{AL} = \frac{\rho_a}{r}.$$

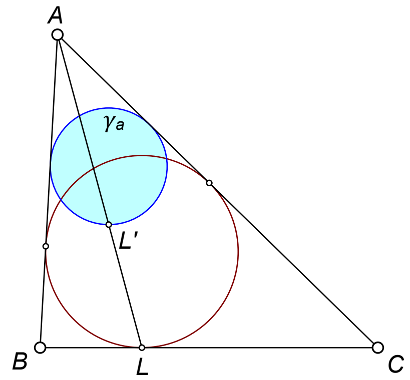


Figure 59.

Proof. This follows from the fact that L and L' are corresponding points in the homothety with center A that maps γ_a into the incircle. □

Theorem 4.15 (Length of AL')**.** *We have (Figure 60)*

$$AL' = \left(1 - \frac{rt}{p-a}\right)\sqrt{\frac{(p-a)[ap-(b-c)^2]}{a}}.$$

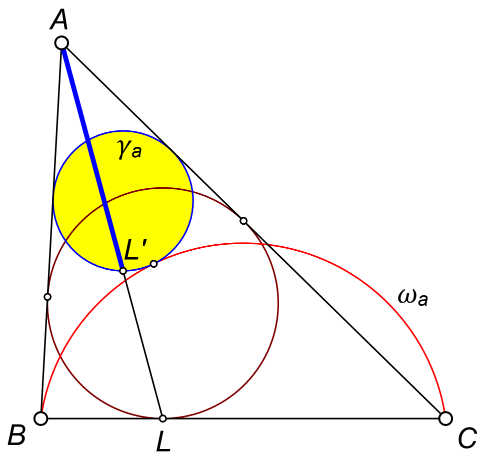


Figure 60.

Proof. Since L is the point where the incircle touches BC, AL is a Gergonne cevian of $\triangle ABC$. The length of a Gergonne cevian is known. From Property 3.1.3 in [26], we have

$$AL = \sqrt{\frac{(p-a)[ap-(b-c)^2]}{a}}. \tag{11}$$

Circle γ_a and the incircle are homothetic, with A being the center of the homothety. Since L' and L are corresponding points of the homothety, we have

$$\frac{AL'}{AL} = \frac{\rho_a}{r}.$$

Thus, $AL' = (\rho_a/r)\cdot AL$. Combining this with the value of ρ_a/r from Corollary 4.7 gives us our result. □

Lemma 4.16. *We have $AK = p - a - rt$.*

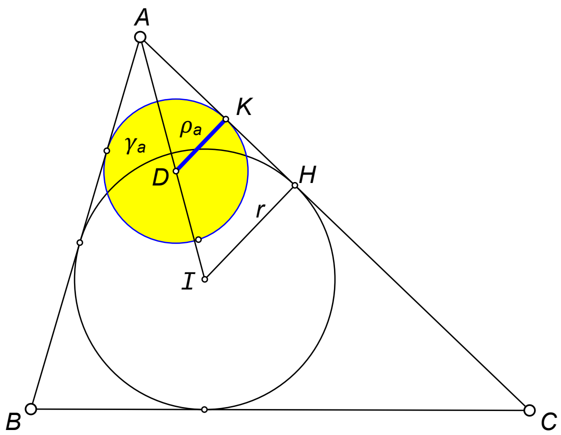


FIGURE 61.

Proof. It is well known that $AH = p - a$ (Figure 61). From Corollary 4.5, we have $HK = rt$. Thus, $AK = AH - HK = p - a - rt$. □

Theorem 4.17 (Length of AX). *We have (Figure 62).*

$$AX = \frac{(p-a-rt)\sqrt{a(p-a)}}{\sqrt{ap-(b-c)^2}}.$$

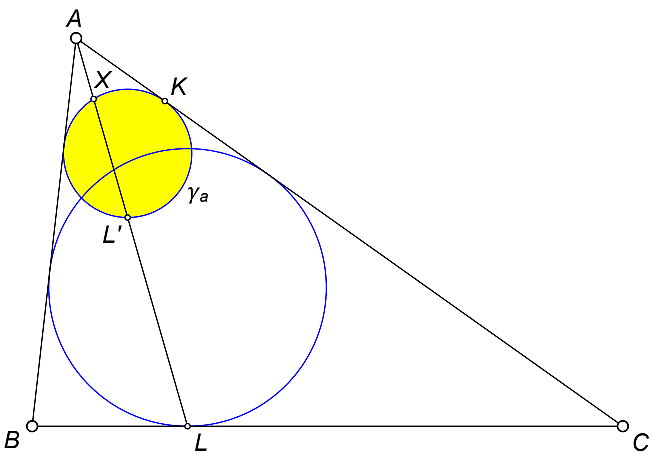


FIGURE 62.

Proof. Since AXL' is a secant to γ_a and AK is a tangent, we have $AX \cdot AL' = (AK)^2$. From Lemma 4.16,

$$AK = p - a - rt.$$

From Theorem 4.15, we have

$$AL' = \left(\frac{p-a-rt}{p-a}\right)\sqrt{\frac{(p-a)[ap-(b-c)^2]}{a}}.$$

So

$$\begin{aligned} AX &= \frac{(AK)^2}{AL'} \\ &= \frac{(p-a-rt)(p-a)}{\sqrt{\frac{(p-a)[ap-(b-c)^2]}{a}}} \\ &= \frac{(p-a-rt)\sqrt{a(p-a)}}{\sqrt{ap-(b-c)^2}}. \end{aligned}$$

□

Corollary 4.18. *We have*

$$\frac{AX}{AL'} = \frac{a(p-a)}{ap-(b-c)^2}.$$

5. Barycentric Coordinates

In this section, we will find the barycentric coordinates for various points associated with our configuration.

Theorem 5.1 (Coordinates for D). *The barycentric coordinates for D are*

$$D = \Big(ap(p-a) + (b+c)t\Delta) : bp(p-a) - bt\Delta : cp(p-a) - ct\Delta\Big)$$

where Δ denotes the area of $\triangle ABC$, p denotes the semiperimeter, and $t = \tan\frac{\theta}{4}$.

Proof. Let y be the distance between D and BC. Summing the areas of triangles DBC, DCA and DAB we obtain

$$ay + b\rho_a + c\rho_a = 2\Delta.$$

Thus,

$$ay = 2\Delta - (b+c)\rho_a.$$

Letting $[XYZ]$ denote the area of $\triangle XYZ$, we find that the barycentric coordinates for D are therefore

$$\begin{aligned} D &= \Big([DBC] : [DCA] : [DAB]\Big) = (ay : b\rho_a : c\rho_a) \\ &= (2\Delta - (b+c)\rho_a : b\rho_a : c\rho_a) \\ &= \left(\frac{2\Delta}{\rho_a} - (b+c) : b : c\right). \end{aligned}$$

Replacing ρ_a by its value given by Corollary 4.7, we get

$$\begin{aligned} D &= \left(\frac{2\Delta}{r\left(1 - \frac{rt}{p-a}\right)} - (b+c) : b : c\right) \\ &= \left(\frac{2(p-a)\Delta}{r\,(p-a-rt)} - (b+c) : b : c\right) \\ &= \Big((b+c-a)\Delta - (b+c)r(p-a-rt) : br(p-a-rt) : cr(p-a-rt)\Big). \end{aligned}$$

Replacing r by Δ/p, then multiplying all coordinates by p^2/Δ gives

$$D = \Big(ap(b+c-p) + (b+c)t\Delta : bp(p-a) - bt\Delta : cp(p-a) - ct\Delta\Big).$$

Finally, noting that $b+c-p = p-a$, gives the desired result. $\square$

Theorem 5.2 (Coordinates for O_a). *The barycentric coordinates for O_a are*

$$O_a = \left(-a^2 : S_c + S\cot\phi : S_b + S\cot\phi\right).$$

where $\phi = 90^\circ - \theta/2$, $S = 2\Delta$, $S_b = (c^2 + a^2 - b^2)/2$, and $S_c = (a^2 + b^2 - c^2)/2$.

Proof. The result follows from Conway's Formula [40, p. 34]. $\square$

Theorem 5.3 (Coordinates for T)**.** *The barycentric coordinates for T are $(T_x : T_y : T_z)$ where*

$$\begin{aligned}
T_x &= 2a\sin\frac{\theta}{4}\left(au\cos\frac{\theta}{2}+(b+c)u+2aS\sin\frac{\theta}{2}\right),\\
T_y &= -u\left(2\left(a^2-bc-c^2\right)\cos\frac{\theta}{2}+a^2+2ab-(b+c)^2\right)\sin\frac{\theta}{4}\\
&\quad+2S\left(a^2+bc-c^2\right)\cos\frac{3\theta}{4}+2bS(2a+b-c)\cos\frac{\theta}{4},\\
T_z &= -u\left(2\left(a^2-bc-b^2\right)\cos\frac{\theta}{2}+a^2+2ac-(b+c)^2\right)\sin\frac{\theta}{4}\\
&\quad+2S\left(a^2+bc-b^2\right)\cos\frac{3\theta}{4}+2cS(2a-b+c)\cos\frac{\theta}{4},
\end{aligned}$$

where $S = 2\Delta$ and $u = a^2 - (b-c)^2$.

Proof. The barycentric coordinates for D were found in Theorem 5.1. This can be simplified to $D = (D_x : D_y : D_z)$ where

$$\begin{aligned}
D_x &= a^3 - a(b+c)^2 - 2S(b+c)t,\\
D_y &= -b\left(-a^2+b^2+2bc+c^2-2St\right),\\
D_z &= -c\left(-a^2+b^2+2bc+c^2-2St\right)
\end{aligned}$$

by using the substitutions $r = S/(a+b+c)$, $p = (a+b+c)/2$, and $\Delta = S/2$.

The barycentric coordinates for O_a were found in Theorem 5.2, namely

$$O_a = \left(-a^2 : S_c + S\cot\phi : S_b + S\cot\phi\right)$$

where $\phi = 90^\circ - \theta/2$, $S_b = (c^2+a^2-b^2)/2$, and $S_c = (a^2+b^2-c^2)/2$.

From Corollary 4.8, we have

$$\rho_a = \frac{S - 2(p-b)(p-c)t}{2p}.$$

The radius of ω_a was found in Theorem 4.11, namely

$$R_a = \frac{a}{2}\csc\frac{\theta}{2}.$$

The touch point T divides the segment DO_a in the ratio $\rho_a : R_a$. This fact allows us to use Mathematica to find the barycentric coordinates for T from the known barycentric coordinates for D and O_a. □

6. Properties of Three Ajima Circles

Let ω_a, ω_b, and ω_c, be arcs of angular measure θ erected internally on the sides of $\triangle ABC$. Let γ_a be the circle inscribed in $\angle BAC$ and tangent externally to ω_a. Define γ_b and γ_c similarly. The three circles, γ_a, γ_b and γ_c will be called a *general triad of circles* associated with $\triangle ABC$ (Figure 63).

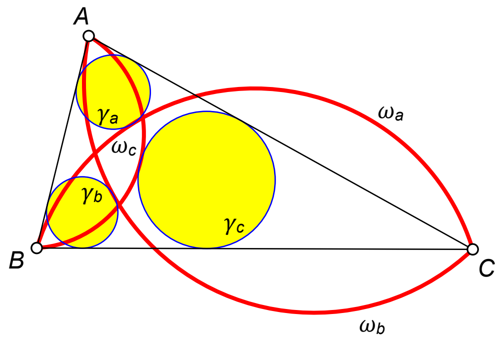


FIGURE 63. general triad of circles

For the remainder of this paper, we will assume that the three circles γ_a, γ_b and γ_c all lie inside $\triangle ABC$. An equivalent condition is that all angles of $\triangle ABC$ have measure less than $180° - \frac{\theta}{2}$.

Theorem 6.1. *The common external tangents to any pair of circles in a general triad are congruent (Figure 64). The common length is $2r \tan \frac{\theta}{4}$.*

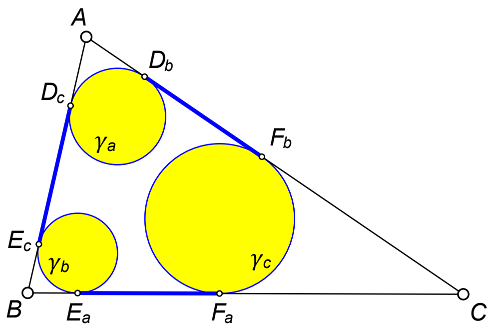


FIGURE 64. blue lines are congruent

Proof. The common length is twice KH (Figure 52) whose value is given by Corollary 4.5. □

Note. The theorem remains true if some or all of the yellow circles are outside of $\triangle ABC$ as shown in Figure 65.

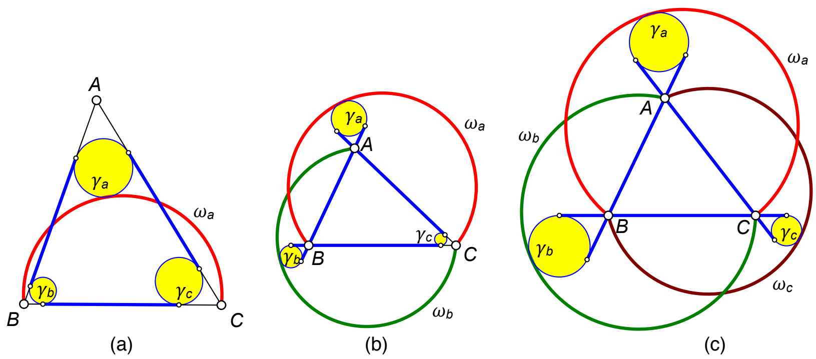


FIGURE 65. blue lines are congruent

Theorem 6.2. *Let M_a, M_b, and M_c be the midpoints of the common tangents (lying along the sides of $\triangle ABC$) to a general triad of circles associated with that triangle. Then M_a, M_b, and M_c are the touch points of the incircle of $\triangle ABC$ with the sides of the triangle (Figure 66).*

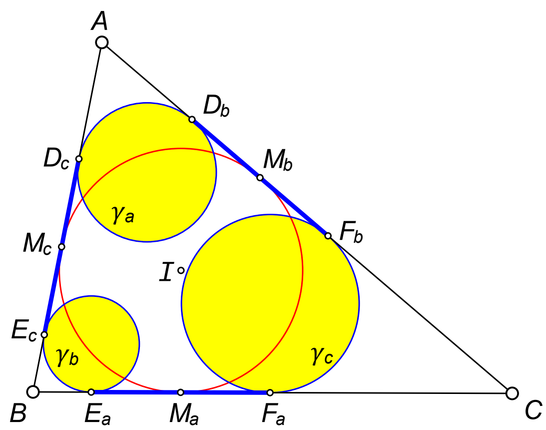


FIGURE 66.

Proof. This follows from Corollary 4.5. □

Theorem 6.3. *Let γ_a, γ_b, and γ_c be a general triad of circles associated with triangle $\triangle ABC$. Let M_a, M_b, and M_c be the points where the incircle of $\triangle ABC$ touches the sides. Then the radical axis of γ_b, and γ_c is AM_a (Figure 67).*

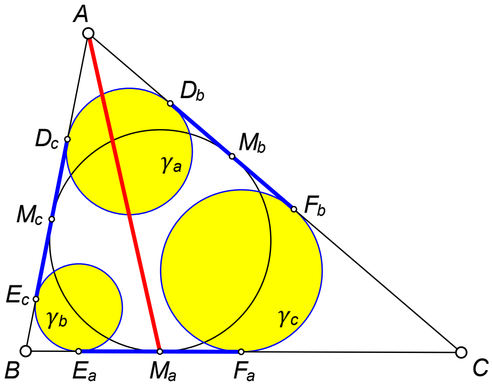


FIGURE 67.

Proof. From Theorem 6.2, $M_aE_a = M_aF_a$. Thus, the tangents from M_a to γ_b and γ_c are equal. Since $AD_c = AD_b$ and $D_cE_c = D_bF_b$ (Theorem 6.1), this means $AE_c = AF_b$. Hence the tangents from A to γ_b and γ_c are equal. The radical axis of circles γ_b and γ_c is the locus of points such that the lengths of the tangents to the two circles from that point are equal. The radical axis of two circles is a straight line. Therefore, the radical axis of circles γ_b and γ_c is AM_a, the Gergonne cevian from A. □

Theorem 6.4. *Let γ_a, γ_b, and γ_c be a general triad of circles associated with triangle $\triangle ABC$. Then the radical center of the three circles of the triad is the Gergonne point of $\triangle ABC$ (Figure 68).*

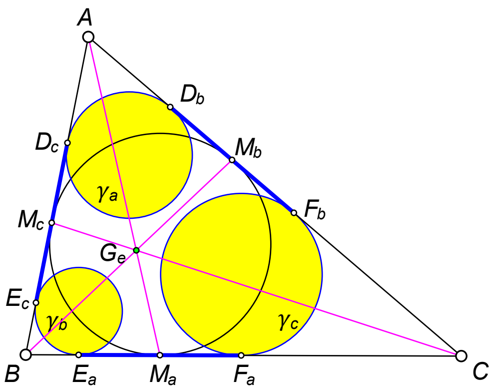


FIGURE 68.

Proof. By Theorem 6.3, the radical axis of circles γ_b and γ_c is AM_a, the Gergonne cevian from A. Similarly, the radical axis of circles γ_a and γ_c is the Gergonne cevian from B and the radical axis of circles γ_a and γ_b is the Gergonne cevian from C. Hence, the radical center of the general triad of circles is the intersection point of the three Gergonne cevians, namely, the Gergonne point of $\triangle ABC$. □

Theorem 6.5. *The six points of contact of a general triad of circles lie on a circle with center I, the incenter of $\triangle ABC$ (Figure 69).*

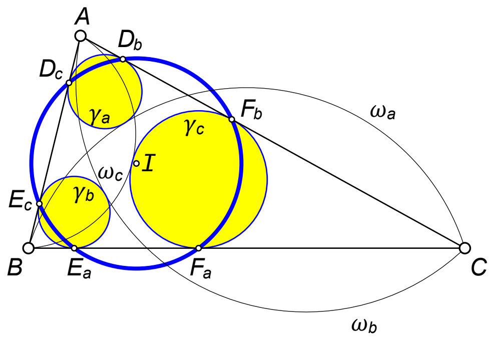


Figure 69. touch points are concyclic

Proof. This follows from Theorem 4.3 from which we can deduce that

$$ID_b = ID_c = IE_a = IE_c = IF_a = IF_b = r\sec\frac{\theta}{4}. \qquad \square$$

Theorem 6.6. *Let the centers of ω_a, ω_b, and ω_b, be O_a, O_b, and O_c, respectively. Then AO_a, BO_b, and CO_c are concurrent (Figure 70).*

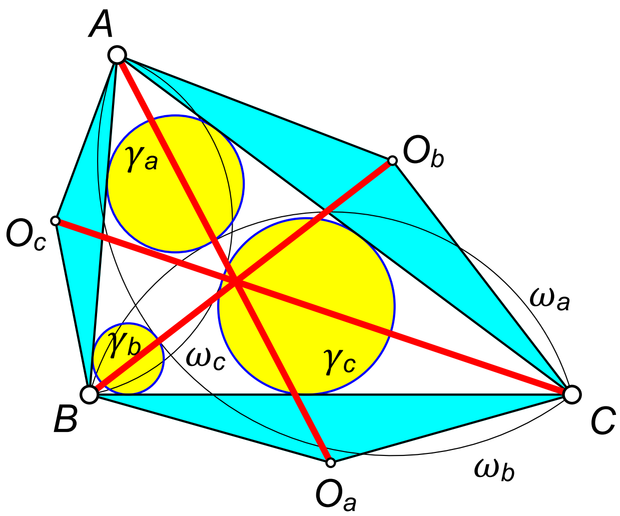


Figure 70. red lines are concurrent

Proof. Note that isosceles triangles BCO_a, CAO_b, and ABO_c are similar. Therefore AO_a, BO_b, and CO_c are concurrent by Jacobi's Theorem [38]. $\square$

Theorem 6.7 (Paasche Analog)**.** *Let γ_a, γ_b, and γ_c be a general triad of circles associated with triangle $\triangle ABC$. Let T_a, T_b, and T_c be the points where they touch the three arcs having the same angular measure (Figure 71). Then AT_a, BT_b, and CT_c are concurrent.*

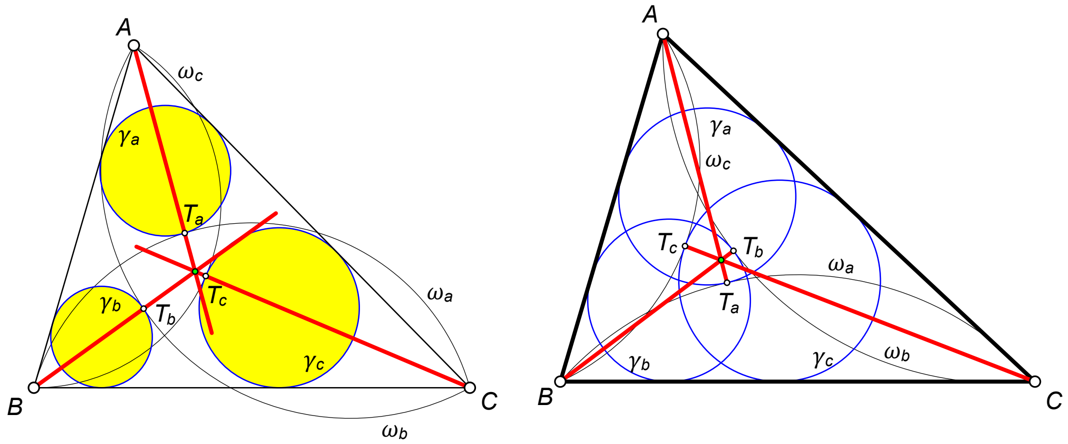


FIGURE 71. red lines are concurrent

Proof. The barycentric coordinates for T_a were found in Theorem 5.3. The barycentric coordinates for A are $(1 : 0 : 0)$. We can thus find the equation of the line AT_a using formula (3) from [15]. Similarly, we can find the equations for the lines BT_b and CT_c. Then, using MATHEMATICA, we can use the condition that three lines are concurrent (formula (6) from [15]) to prove that AT_a, BT_b and CT_c are concurrent. □

We call this theorem the Paasche Analog because when $\theta = 180°$, the point of concurrence is the Paasche point of the triangle [19].

Open Question 1. *Is there a purely geometric proof for Theorem 6.7?*

The coordinates for the point of concurrence are complicated and we do not give them here. However, we did find the following interesting result.

Theorem 6.8. *When $\theta = 120°$, the point of concurrence of AT_a, BT_b, and CT_c is the isogonal conjugate of X_{7005}. When $\theta = 240°$, the point of concurrence of AT_a, BT_b, and CT_c is X_{14358}.*

7. Some Metric Identities

Throughout this section, we will let

$$t = \tan \frac{\theta}{4}$$

and

$$\mathbb{W} = \frac{4R + r}{p}.$$

The following three identities were given in [29, Lemma 3] and we will need them here as well.

Lemma 7.1. *Let A, B, and C be the angles of a triangle with inradius r, circumradius R, and semiperimeter p. Then*

$$\tan \frac{A}{2} + \tan \frac{B}{2} + \tan \frac{C}{2} = \mathbb{W}.$$

Lemma 7.2. *Let A, B, and C be the angles of a triangle. Then*

$$\tan \frac{A}{2} \tan \frac{B}{2} + \tan \frac{B}{2} \tan \frac{C}{2} + \tan \frac{C}{2} \tan \frac{A}{2} = 1.$$

Lemma 7.3. *Let A, B, and C be the angles of a triangle with inradius r and semiperimeter p. Then*

$$\tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2} = \frac{r}{p}.$$

From Theorem 4.6,

$$\rho_a = r \left(1 - t \tan \frac{A}{2}\right),$$

so

$$r - \rho_a = rt \tan \frac{A}{2} \tag{12}$$

with similar formulas for $r - \rho_b$ and $r - \rho_c$. Also,

$$\tan \frac{A}{2} = \frac{r - \rho_a}{rt} \tag{13}$$

with similar formulas for $\tan \frac{B}{2}$ and $\tan \frac{C}{2}$. Using equation (12) gives us the following corollary to these lemmas.

Corollary 7.4. *For a general triad of circles associated with $\triangle ABC$, we have*

$$\sum (r - \rho_a) = rt\mathbb{W}, \tag{14}$$

$$\sum (r - \rho_a)(r - \rho_b) = r^2 t^2, \tag{15}$$

$$\prod (r - \rho_a) = \frac{r^4 t^3}{p}. \tag{16}$$

Theorem 7.5. *For a general triad of circles associated with $\triangle ABC$, we have*

$$\rho_a + \rho_b + \rho_c = 3r - rt\mathbb{W}.$$

Proof. This follows immediately from equation (14). □

When $\theta = 180°$, the arcs become semicircles, $t = 1$, and this result agrees with formula (6) in [29].

Theorem 7.6. *For a general triad of circles associated with $\triangle ABC$, we have*

$$3r^2 - 2r\sum \rho_a + \sum \rho_a\rho_b = r^2t^2.$$

Proof. Expanding the left side of equation (15) gives the desired result. $\square$

Theorem 7.7. *For a general triad of circles associated with $\triangle ABC$, we have*

$$\rho_a\rho_b + \rho_b\rho_c + \rho_c\rho_a = r^2\left(t^2 - 2t\mathbb{W} + 3\right).$$

Proof. From Theorem 7.6, we have

$$3r^2 - 2r\sum \rho_a + \sum \rho_a\rho_b = r^2t^2.$$

Using Theorem 7.5, we get

$$3r^2 - 2r\left(3r - rt\mathbb{W}\right) + \sum \rho_a\rho_b = r^2t^2.$$

Thus,

$$\sum \rho_a\rho_b = r^2t^2 + 2r\left(3r - rt\mathbb{W}\right) - 3r^2$$

which simplifies to

$$\sum \rho_a\rho_b = r^2t^2 - 2r^2t\mathbb{W} + 3r^2$$

as desired. $\square$

When $\theta = 180°$, the arcs become semicircles and this result agrees with formula (7) in [29].

Theorem 7.8. *We have*

$$\rho_a^2 + \rho_b^2 + \rho_c^2 = r^2\left[3 - 2t\mathbb{W} + \left(\mathbb{W}^2 - 2\right)t^2\right]$$

Proof. Using the identity

$$\left(\sum \rho_a\right)^2 = \sum \rho_a^2 + 2\sum \rho_a\rho_b,$$

we find that

$$\rho_a^2 + \rho_b^2 + \rho_c^2 = \left(3r - rt\mathbb{W}\right)^2 - 2r^2\left(t^2 - 2t\mathbb{W} + 3\right).$$

Simplifying gives

$$\rho_a^2 + \rho_b^2 + \rho_c^2 = r^2\left[3 - 2t\mathbb{W} + \left(\mathbb{W}^2 - 2\right)t^2\right]$$

which is the desired result. $\square$

When $\theta = 180°$, the arcs become semicircles and this result agrees with formula (8) in [29].

Theorem 7.9. *For a general triad of circles associated with $\triangle ABC$, we have*

$$\rho_a\rho_b\rho_c = r^3\left(1 - t\mathbb{W} + t^2 - \frac{r}{p}t^3\right).$$

Proof. Start with equation (16). Expand and use Theorems 7.5 and 7.7 to substitute known values for $\sum \rho_a$ and $\sum \rho_a\rho_b$. Solving for $\rho_a\rho_b\rho_c$ then gives the desired formula. $\square$

The following result was found empirically using the program "OK Geometry".[3]

Theorem 7.10. *We have*

$$a^2\rho_a^2(2r-\rho_a)^2+16r(r-\rho_a)(rR-rR_a-\rho_aR)(rR-rRa+\rho_aR_a)=0.$$

Proof. Starting with the left side of the equation, we make the following substitutions, in succession.

$$\begin{aligned}
\rho_a &= r\left(1-\frac{r}{p-a}\tan\frac{\theta}{4}\right)\\
R &= \frac{abc}{4\Delta}\\
r &= \frac{\Delta}{p}\\
\Delta &= \sqrt{p(p-a)(p-b)(p-c)}\\
p &= \frac{a+b+c}{2}\\
R_a &= \frac{a}{2\cos(90^\circ-\frac{\theta}{2})}
\end{aligned}$$

Simplifying the resulting expression using Mathematica, we find that the expression is equal to 0. ☐

In some special cases, this formula can be simplified.

Theorem 7.11. *If* $\theta = 360^\circ - 4A$*, then*

$$\rho_a = \frac{2rR_a}{R+2R_a}.$$

Proof. The proof is the same as the proof of Theorem 7.10. ☐

For a fixed θ, we can find a relationship between r, R, R_a and ρ_a, not involving a.

Theorem 7.12. *We have*

$$\frac{R_a}{rR} = \frac{(r-\rho_a)(1+t^2)}{(r-\rho_a)^2+r^2t^2}. \tag{17}$$

Proof. This follows by eliminating $\tan(A/2)$ from equations (8) and (10). The expression $\sin A$ is expressed in terms of $\tan(A/2)$ using Lemma 4.10. ☐

Solving for t^2 in equation (17) gives us the following.

Theorem 7.13. *We have*

$$t^2 = \frac{(r-\rho_a)(\rho_aR_a+rR-rR_a)}{r(R\rho_a+rR_a-rR)}.$$

[3]OK Geometry is a tool for analyzing dynamic geometric constructions, developed by Zlatan Magajna which can be freely downloaded from `https://www.ok-geometry.com/`.

8. Apollonius Circles of the Three Ajima Circles

A circle that is tangent to three given circles is called an *Apollonius circle* of those three circles.

If all three circles lie inside an Apollonius circle, then the Apollonius circle is called the *outer Apollonius circle* of the three circles. The outer Apollonius circle surrounds the three circles and is internally tangent to all three.

If all three circles lie outside an Apollonius circle, then the Apollonius circle is called the *inner Apollonius circle* of the three circles. The inner Apollonius circle will either be internally tangent to the three given circles or it will be externally tangent to all the circles. Figure 72 shows various configurations. In each case, the red circle is the inner Apollonius circle of the three blue circles.

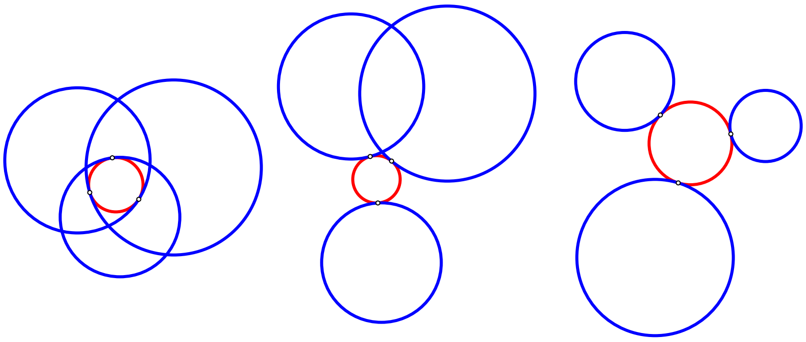


FIGURE 72. inner Apollonius circle of three circles

We will be looking at the inner and outer Apollonius circles of a general triad of circles associated with $\triangle ABC$. But first, let us review some known facts about tangent circles.

Lemma 8.1. *Let $U(r_1)$ and $V(r_2)$ be two circles in the plane. Let S be a center of similarity of the two circles (Figure 73). Then*

$$\frac{US}{SV} = \frac{r_1}{r_2}.$$

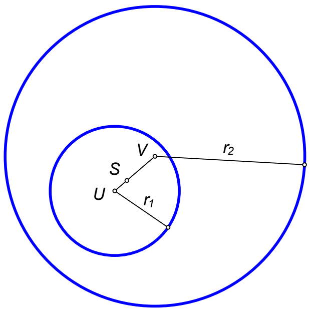


FIGURE 73.

Proof. The line of centers of two circles points passes through the center of similitude. So S lies on UV. In a similarity, corresponding distances in two similar figures are in proportion to their ratio of similitude. Their ratio of similitude is the ratio of their radii, namely r_1/r_2. So $SU/SV = r_1/r_2$. $\square$

When we say that a circle is inscribed in an angle ABC, we mean that the circle is tangent to the rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$.

The following result comes from [25, Theorem 2].

Lemma 8.2. *Let C_a be an arbitrary circle inscribed in $\angle BAC$ of $\triangle ABC$. Let C_b be an arbitrary circle inscribed in $\angle CBA$. Let C_c be an arbitrary circle inscribed in $\angle ACB$. Let S be the inner (respectively outer) Apollonius circle of C_a, C_b, and C_c. Let T_a be the point where C_a touches (S). Define T_b and T_c similarly. Then AT_a, BT_b, and CT_c are concurrent at a point P (Figure 74). The point P is the internal (external) center of similitude of the incircle of $\triangle ABC$ and circle (S).*

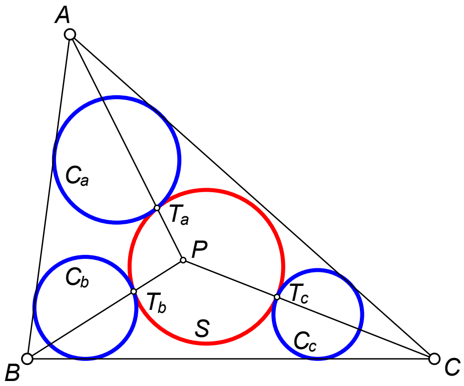


Figure 74.

The following lemma comes from [7, p. 85].

Lemma 8.3. *If two circles touch two others, then the radical axis of either pair passes through a center of similitude of the other pair (Figure 75).*

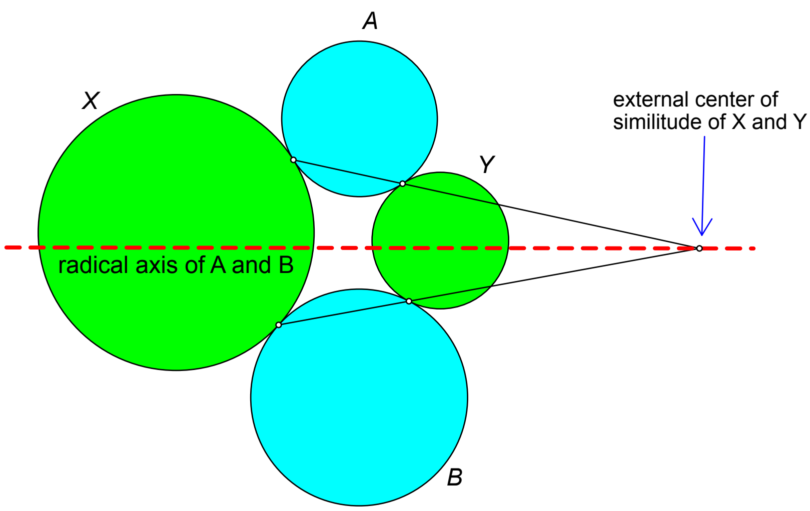


Figure 75.

The following lemma comes from Gergonne's construction of Apollonius circles. (See [11, pp. 159–160].)

Lemma 8.4. *Let (O_1), (O_2), and (O_3) be three circles in the plane. Let C_i be the inner Apollonius circle of the circles (O_1), (O_2), and (O_3). Let C_o be the outer Apollonius circle of the circles (O_1), (O_2), and (O_3). Let U_1 be the point where C_i touches (O_1). Define U_2 and U_3 similarly. Let V_1 be the point where C_o touches (O_1). Define V_2 and V_3 similarly. Then V_1U_1, V_2U_2, and V_3U_3 are concurrent at the radical center, R, of (O_1), (O_2), and (O_3) (Figure 76).*

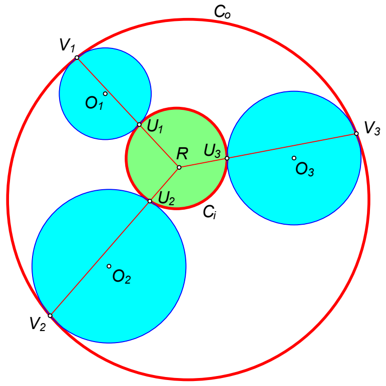


FIGURE 76.

Theorem 8.5. *Let C_1, C_2, and C_3 be three circles as shown in Figure 77. Let $U(\rho_i)$ and $V(\rho_o)$ be the inner and outer Apollonius circles of C_1, C_2, and C_3, respectively. Let S be the radical center of the three circles. Then S lies on UV and*

$$\frac{SU}{SV} = \frac{\rho_i}{\rho_o}.$$

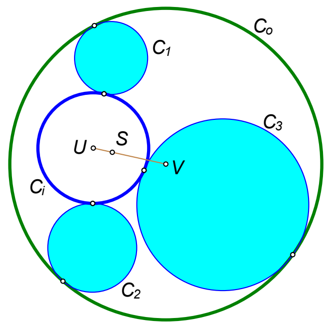


FIGURE 77. $SU/SV = \rho_i/\rho_o$

Proof. Circles C_1 and C_2 each touch circles C_i and C_o. By Lemma 8.3, the radical axis of C_1 and C_2 passes through a center of similarity, S^*, of C_i and C_o. Similarly, the radical axis of C_2 and C_3 passes through S^*. These two radical axes meet at S, so $S = S^*$.

Note that C_i and C_o are two circles with center of similarity S. By Lemma 8.1, S lies on UV and $SU/SV = \rho_i/\rho_o$. □

The following results were found via complex calculations carried out with Mathematica. The details are omitted.

Theorem 8.6 (Coordinates for U_a)**.** *Let γ_a, γ_b, and γ_c be a general triad of circles associated with $\triangle ABC$. Let C_i be the inner Apollonius circle of the circles in the triad. Let U_a be the point where C_i touches γ_a. Then the barycentric coordinates for U_a are $(x : y : z)$ where*

$$\begin{aligned} x &= 2a(p-b)(p-c)t \\ y &= (p-c)[S-2(p-b)(p-c)t] \\ z &= (p-b)[S-2(p-b)(p-c)t] \end{aligned}$$

and where p is the semiperimeter of $\triangle ABC$, S is twice the area, and $t = \tan(\theta/4)$.

The coordinates for U_b and U_c are similar.

Theorem 8.7 (Coordinates for U)**.** *Let γ_a, γ_b, and γ_c be a general triad of circles associated with $\triangle ABC$. Let C_i be the inner Apollonius circle of the circles in the triad. Let U be the center of C_i. Then the barycentric coordinates for U are $(X : Y : Z)$ where*

$$\begin{aligned} X &= \left(-2a^3 + a^2(b+c) + (b-c)^2(b+c)\right)t - 2aS \\ Y &= \left(a^3 - a^2c + a\left(b^2-c^2\right) + c^3 + b^2c - 2b^3\right)t - 2bS \\ Z &= \left(a^3 - a^2b + a\left(c^2-b^2\right) + b^3 + bc^2 - 2c^3\right)t - 2cS \end{aligned}$$

and where S is twice the area of $\triangle ABC$ and $t = \tan(\theta/4)$.

Theorem 8.8 (Coordinates for V_a)**.** *Let γ_a, γ_b, and γ_c be a general triad of circles associated with $\triangle ABC$. Let C_o be the outer Apollonius circle of the circles in the triad. Let V_a be the point where C_o touches γ_a. Then the barycentric coordinates for V_a are $(x : y : z)$ where*

$$\begin{aligned} x &= 2(p-b)(p-c)[2S + a(p-a)t] \\ y &= (p-a)(p-c)[S-2(p-b)(p-c)t] \\ z &= (p-a)(p-b)[S-2(p-b)(p-c)t] \end{aligned}$$

and where p is the semiperimeter of $\triangle ABC$, S is twice the area, and $t = \tan(\theta/4)$.

The coordinates for V_b and V_c are similar.

Theorem 8.9 (Coordinates for V)**.** *Let γ_a, γ_b, and γ_c be a general triad of circles associated with $\triangle ABC$. Let C_o be the outer Apollonius circle of the circles in the triad. Let V be the center of C_o. Then the barycentric coordinates for V are $(X : Y : Z)$ where*

$$\begin{aligned} X &= \left(-2a^3 + a^2(b+c) + (b-c)^2(b+c)\right)t + 6aS \\ Y &= \left(a^3 - a^2c + a\left(b^2-c^2\right) + c^3 + b^2c - 2b^3\right)t + 6bS \\ Z &= \left(a^3 - a^2b + a\left(c^2-b^2\right) + b^3 + bc^2 - 2c^3\right)t + 6cS \end{aligned}$$

and where S is twice the area of $\triangle ABC$ and $t = \tan(\theta/4)$.

Theorem 8.10. *Let γ_a, γ_b, and γ_c be a general triad of circles associated with $\triangle ABC$. Let C_i be the inner Apollonius circle of the circles in the triad. Let U_a be the point where C_i touches γ_a. Define U_b and U_c similarly. Then A, U_a, and G_e are collinear (Figure 78). Similarly, B, U_b, and G_e are collinear; and C, U_c, and G_e are collinear.*

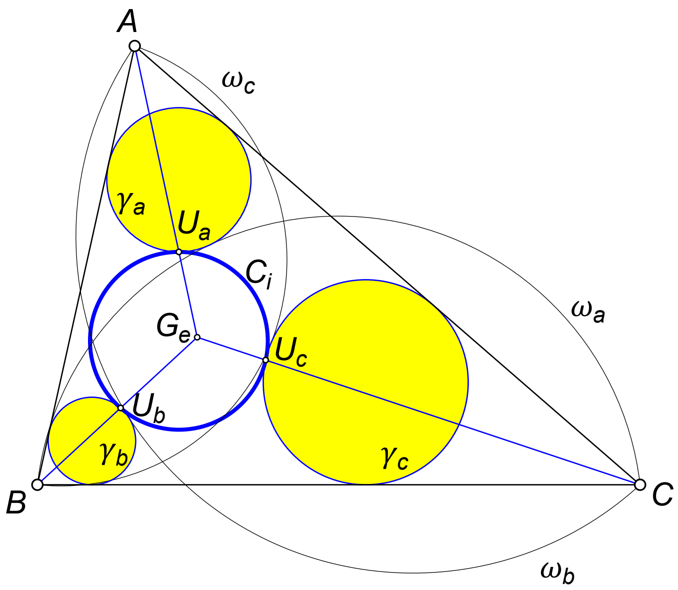


FIGURE 78. lines concur at the Gergonne point

Proof. By symmetry, it suffices to prove that A, U_a, and G_e are collinear. The barycentric coordinates for A are $(1 : 0 : 0)$. The barycentric coordinates for G_e are well known to be

$$G_e = \left(\frac{1}{b+c-a} : \frac{1}{c+a-b} : \frac{1}{a+b-c}\right).$$

The barycentric coordinates for U_a were given in Theorem 8.6. Using these coordinates and the condition for three points to be collinear (formula (4) from [15]), it is straightforward to confirm that A, U_a, and G_e are collinear. $\square$

Open Question 2. *Is there a purely geometric proof for Theorem 8.10?*

Corollary 8.11. *The point we called L' in Section 3 (the intersection of AL with γ_a nearer L) coincides with U_a, the point where the inner Apollonius circle touches γ_a.*

Theorem 8.12. *Let γ_a, γ_b, and γ_c be a general triad of circles associated with $\triangle ABC$. Let C_o be the outer Apollonius circle of the circles in the triad. Let V_a be the point where C_o touches γ_a. Define V_b and V_c similarly. Then A, V_a, and G_e are collinear (Figure 79). Similarly, B, V_b, and G_e are collinear; and C, V_c, and G_e are collinear.*

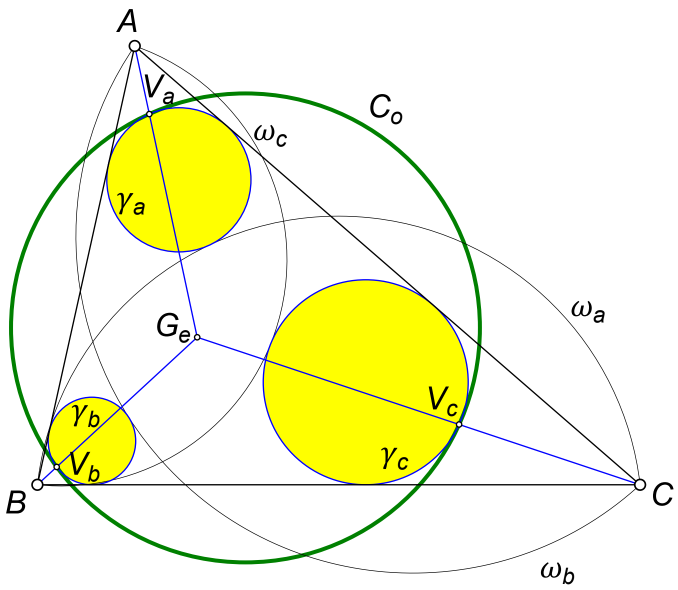


Figure 79. lines concur at the Gergonne point

Proof. By Lemma 8.4, U_aV_a, U_bV_b, and U_cV_c concur at the radical center of γ_a, γ_b, and γ_c. By Theorem 6.4, this radical center is the Gergonne point of the triangle. So V_a lies on G_eU_a. By Theorem 8.10, A, U_a, and G_e are collinear. So A lies on G_eU_a. Since both A and V_a lie on G_eU_a, we see that V_a lies on AU_a. Similarly, V_b lies on BU_b and V_c lies on CU_c. □

Corollary 8.13. *The point we called X in Section 3 (the intersection of AL with γ_a nearer A) coincides with V_a, the point where the outer Apollonius circle touches γ_a.*

Combining Theorems 8.10 and 8.12 lets us state the following result.

Theorem 8.14. *Let γ_a, γ_b, and γ_c be a general triad of circles associated with $\triangle ABC$. Let C_i be the inner Apollonius circle of the circles in the triad. Let C_o be the outer Apollonius circle of the circles in the triad. Let U_a be the point where C_i touches γ_a. Define U_b and U_c similarly. Let V_a be the point where C_o touches γ_a. Define V_b and V_c similarly. Then A, V_a, and U_a are collinear. Similarly, B, V_b, and U_b and C, V_c, and U_c are collinear. The three lines meet at G_e, the Gergonne point of $\triangle ABC$ (Figure 80).*

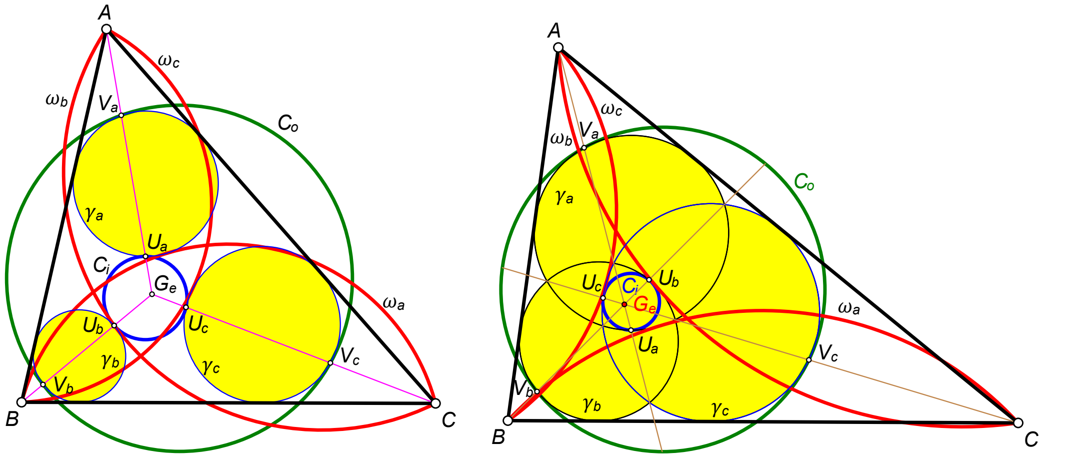


FIGURE 80. lines concur at the Gergonne point

Theorem 8.15. *Let γ_a, γ_b, and γ_c be a general triad of circles associated with $\triangle ABC$. Let C_i be the inner Apollonius circle of the circles in the triad. Let U_a be the point where C_i touches γ_a. Let L be the point where the incircle of $\triangle ABC$ touches BC (Figure 81). Then A, U_a, and L are collinear.*

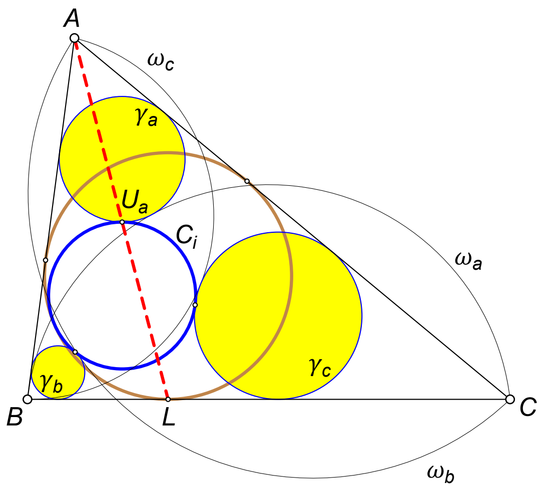


FIGURE 81. A, U_a, and L are collinear

Proof. By Theorem 8.14, AU_a passes through G_e, the Gergonne point of $\triangle ABC$. But by definition, AL also passes through G_e. Thus, AU_a coincides with AL. $\square$

This gives us an easy way to construct the inner Apollonius circle of a general triad of circles. Let the incircle of $\triangle ABC$ touch BC at L. Then AL meets γ_a (closer to L) at U_a. Construct U_b and U_c in the same manner. Then the circumcircle of $\triangle U_aU_bU_c$ is the inner Apollonius circle.

To construct the outer Apollonius circle, find the point V_a where AL meets γ_a (closer to A). Construct V_b and V_c in the same manner. Then the circumcircle of $\triangle V_aV_bV_c$ is the outer Apollonius circle.

Theorem 8.16. *Let γ_a, γ_b, and γ_c be a general triad of circles associated with $\triangle ABC$. Let C_i be the inner Apollonius circle of the circles in the triad. Let U_a be the point where C_i touches γ_a. Let t_a be the tangent to γ_a at U_a. Define t_b and t_c similarly (Figure 82). Then t_a, t_b, and t_c form a triangle homothetic to $\triangle ABC$. The center of the homothety is G_e, the Gergonne point of $\triangle ABC$.*

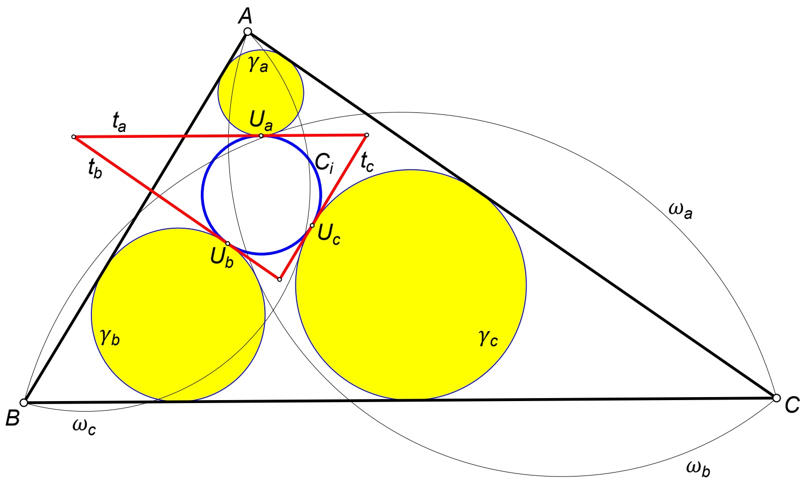


FIGURE 82. red triangle is homothetic to $\triangle ABC$

Proof. By Theorem 3.2, the tangent to γ_a at U_a is parallel to BC. So $t_a \parallel BC$, $t_b \parallel CA$, and $t_c \parallel AB$. Thus, the triangle formed by t_a, t_b, and t_c, is similar to $\triangle ABC$. Let A', B', and C' be the vertices of this triangle. By a well-known theorem [18, Art. 24], this implies that $\triangle ABC$ is homothetic to $\triangle A'B'C'$ (with A mapping to A', B to B', and C to C'). Let L, M, and N be the points where the incircle of $\triangle ABC$ touches BC, CA, and AB (Figure 83).

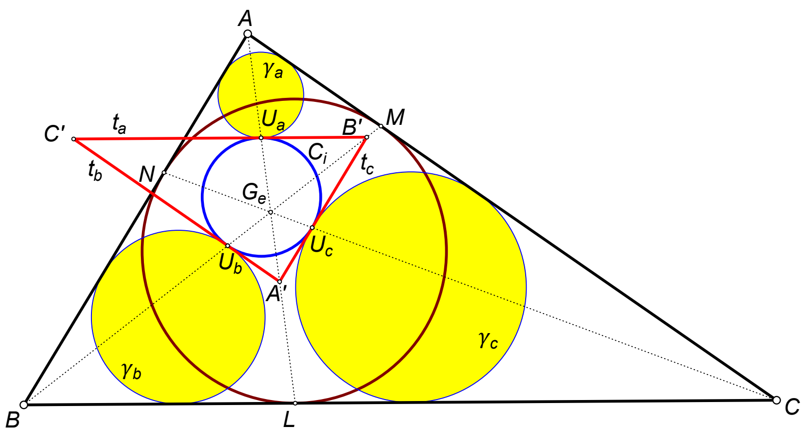


FIGURE 83.

The homothety maps the incircle of $\triangle ABC$ into the incircle of $\triangle A'B'C'$, and the touch points into the touch points, i.e. L maps to U_a, M maps to U_b, and N maps to U_c. Hence, the center of the homothety is the point of concurrence of lines LU_a, MU_b, and NU_c. By Theorem 8.15, the line LU_a coincides with the line AL, the line MU_b coincides with BM, and the line NU_c coincides with line CN. Therefore, the center of the homothety is G_e, the Gergonne point of $\triangle ABC$. ☐

Corollary 8.17. *Let γ_a, γ_b, and γ_c be a general triad of circles associated with $\triangle ABC$. Let C_i be the inner Apollonius circle of the circles in the triad. Then C_i and the incircle of $\triangle ABC$ are homothetic with G_e as the center of the homothety (Figure 84).*

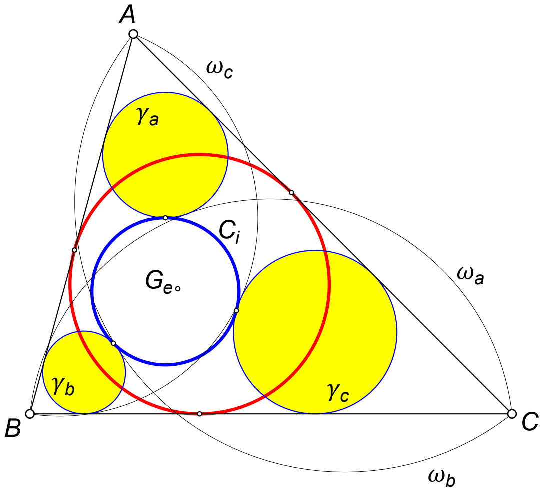


FIGURE 84. G_e is the center of similarity between the incircle and C_i

Theorem 8.18. *For a general triad of circles associated with $\triangle ABC$, let $U(\rho_i)$ be the inner Apollonius circle of γ_a, γ_b, and γ_c (Figure 85). Let G_e be the Gergonne point of $\triangle ABC$. Then*

$$\frac{G_eU}{G_eI} = \frac{\rho_i}{r}.$$

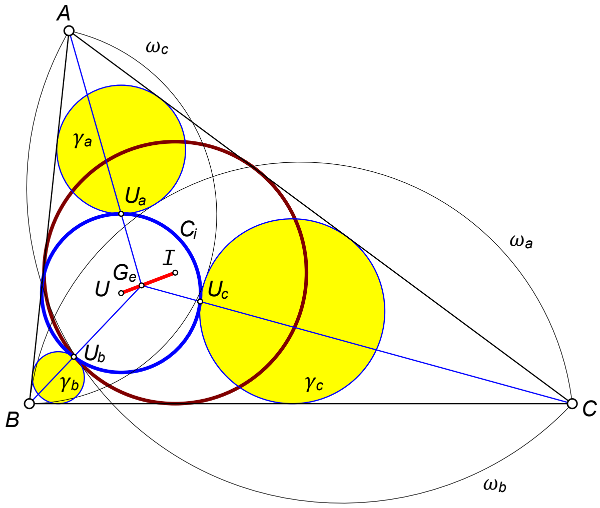


FIGURE 85. $G_eU/G_eI = \rho_i/r$

Proof. Let the touch points of circle (U) with γ_a, γ_b, and γ_c be U_a, U_b, and U_c, respectively. By Lemma 8.2, AU_a, BU_b, and CU_c concur at a point P that is a center of similitude of circle (U) and the incircle, (I). By Theorem 8.14, $P = G_e$. Note that (U) and (I) are two circles with center of similarity G_e. By Lemma 8.1, $G_eU/G_eI = \rho_i/r$. $\square$

Theorem 8.19. *For a general triad of circles associated with $\triangle ABC$, let $V(\rho_o)$ be the outer Apollonius circle of γ_a, γ_b, and γ_c (Figure 86). Let G_e be the Gergonne point of $\triangle ABC$. Then*

$$\frac{G_eV}{G_eI} = \frac{\rho_o}{r}.$$

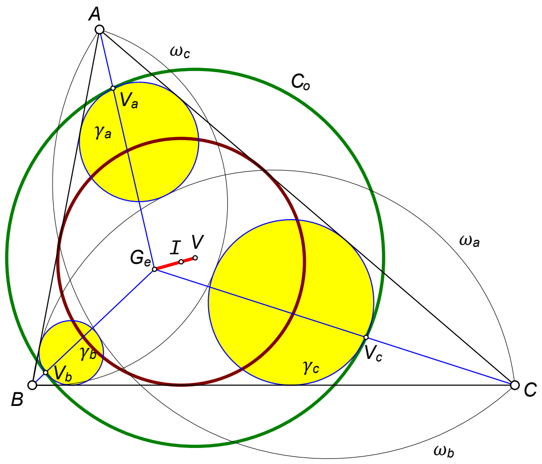


Figure 86. $G_eV/G_eI = \rho_o/r$

Proof. The proof is the same as the proof of Theorem 8.18. ☐

Theorem 8.20. *For a general triad of circles associated with $\triangle ABC$, let $U(\rho_i)$ and $V(\rho_o)$ be the inner and outer Apollonius circles of γ_a, γ_b, and γ_c, respectively. Let G_e be the Gergonne point of $\triangle ABC$. Then*

$$\frac{G_eV}{G_eU} = \frac{\rho_o}{\rho_i}.$$

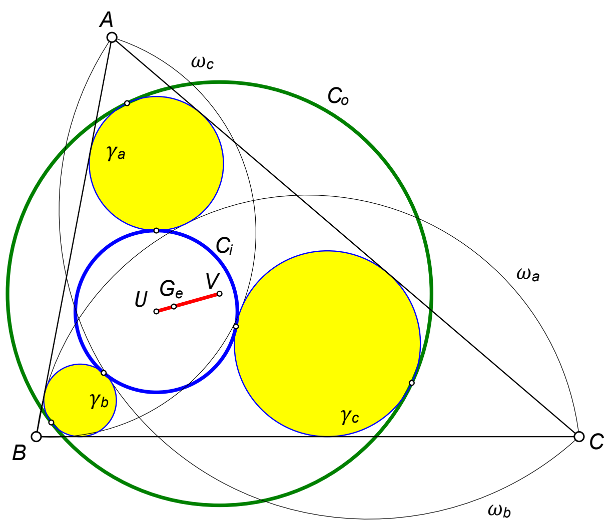


Figure 87. $G_eV/G_eU = \rho_o/\rho_i$

Proof. This follows from Theorem 8.5. It also follows from Theorems 8.18 and 8.19. ☐

Theorem 8.21. *Let γ_a, γ_b, and γ_c be a general triad of circles associated with triangle $\triangle ABC$. A circle externally tangent to each circle of the triad touches γ_a at U_a. Then the tangents from U_a to γ_b and γ_c have the same length (Figure 88).*

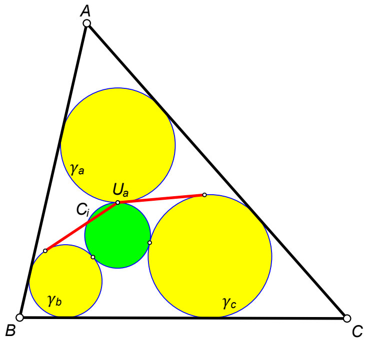


FIGURE 88. red tangent lengths are equal

Proof. By Theorem 8.14, U_a lies on the Gergonne cevian from vertex A. By Theorem 6.3, this Gergonne cevian is the radical axis of circles γ_b and γ_c. Thus, the two tangents have the same length. □

Theorem 8.22 (Miyamoto Analog)**.** *For a general triad of circles associated with $\triangle ABC$, the inner Apollonius circle of γ_a, γ_b, γ_c (blue circle in Figure 89), is internally tangent to the inner Apollonius circle of ω_a, ω_b, ω_c (green circle in Figure 89).*

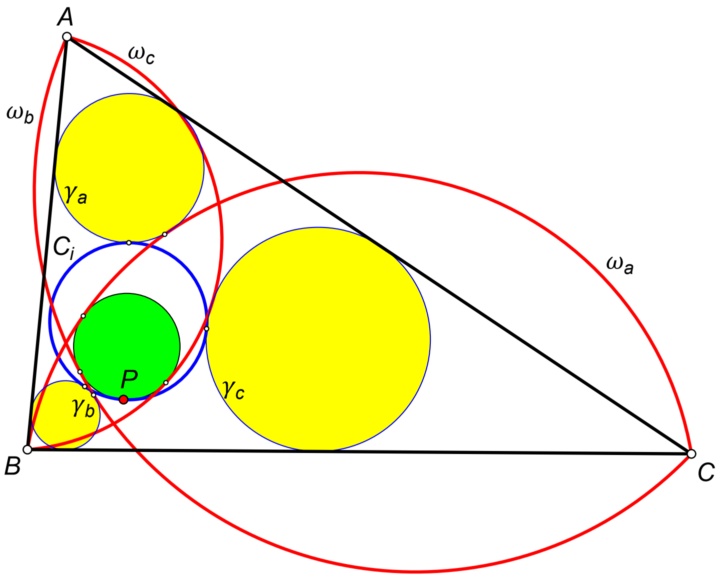


FIGURE 89. green and blue circles touch at P

Proof. This is a special case of the following theorem which is stated in [20]. □

Theorem 8.23 (Miyamoto Generalization). *Let ω_a be any arc erected internally on side BC of $\triangle ABC$. Let γ_a be the circle that is inside $\triangle ABC$, tangent to AB and AC, and tangent externally to ω_a. Define ω_b, ω_c, γ_b, and γ_c similarly. Then the inner Apollonius circle of γ_a, γ_b, γ_c (blue circle in Figure 90), is internally tangent to the inner Apollonius circle of ω_a, ω_b, ω_c (green circle in Figure 90).*

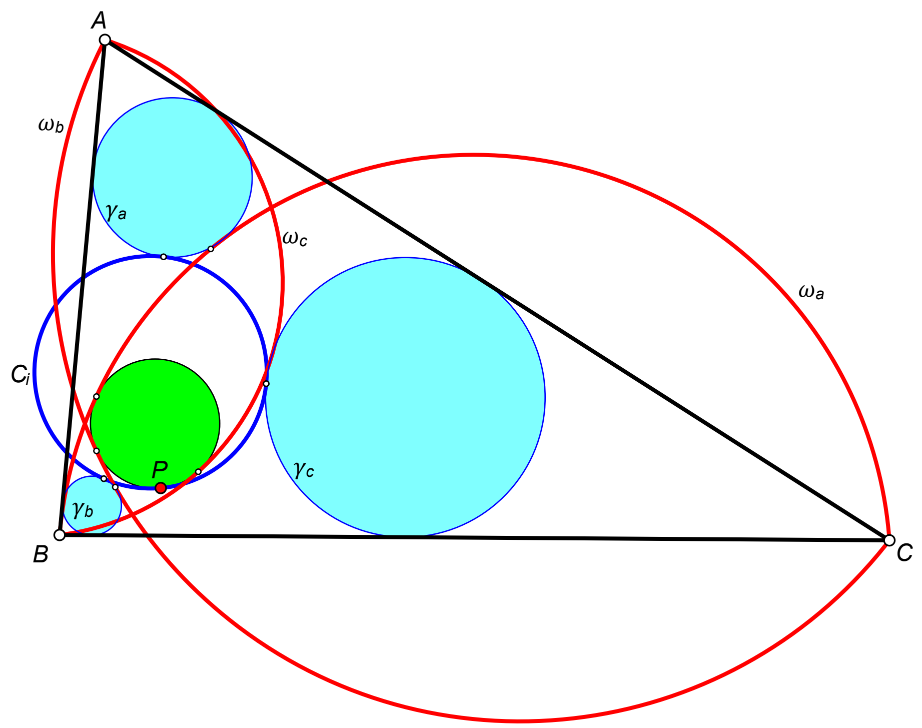


Figure 90. green and blue circles touch at P. Red arcs have different angular measures.

Lemma 8.24. *We have*

$$r\mathbb{W} = \frac{2ab + 2bc + 2ca - a^2 - b^2 - c^2}{2(a+b+c)}.$$

Proof. Recall that $\mathbb{W} = (4R + r)/p$. We use the well-known identities $r = \Delta/p$, $R = abc/(4\Delta)$, and $\Delta = \sqrt{p(p-a)(p-b)(p-c)}$. Then we have

$$\begin{aligned} r\mathbb{W} &= r\left(\frac{4R+r}{p}\right) \\ &= \left(\frac{\Delta}{p}\right)\left(4 \cdot \frac{abc}{4\Delta} + \frac{\Delta}{p}\right) \Big/ p \\ &= \left(\frac{abc}{p} + \frac{\Delta^2}{p^2}\right) \Big/ p \\ &= \frac{1}{p}\left(\frac{abc}{p} + \frac{p(p-a)(p-b)(p-c)}{p^2}\right). \end{aligned}$$

Letting $p = (a+b+c)/2$ and simplifying, gives

$$r\mathbb{W} = \frac{2ab + 2bc + 2ca - a^2 - b^2 - c^2}{2(a+b+c)}. \qquad \square$$

Lemma 8.25 (Length of AG_e)**.** *We have*

$$AG_e = \frac{(p-a)\sqrt{a(p-a)[ap-(b-c)^2]}}{pr\mathbb{W}}.$$

Proof. The distance from a vertex of a triangle to its Gergonne point is known. From Property 2.1.1 in [26], we have

$$AG_e = \frac{(b+c-a)\sqrt{a(b+c-a)[2ap-2(b-c)^2]}}{2ab+2bc+2ca-a^2-b^2-c^2}. \tag{18}$$

From Lemma 8.24, this can be written as

$$AG_e = \frac{(b+c-a)\sqrt{a(b+c-a)[2ap-2(b-c)^2]}}{4pr\mathbb{W}}.$$

Noting that $b+c-a = 2(p-a)$, gives us our result. □

Lemma 8.26. *Let the touch points of the incircle of $\triangle ABC$ with its sides be L, M, and N, as shown in Figure 91. Then*

$$\frac{LG_e}{AG_e} = \frac{(p-b)(p-c)}{a(p-a)}.$$

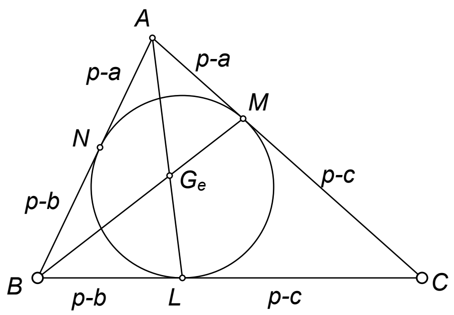


FIGURE 91.

Proof. By definition, G_e is the intersection of AL and BM. Applying Menelaus' Theorem to $\triangle ALC$ with transversal BM gives

$$AG_e \cdot BL \cdot CM = LG_e \cdot BC \cdot AM$$

or

$$(AG_e)(p-b)(p-c) = (LG_e)(a)(p-a)$$

which is equivalent to our desired result. □

Corollary 8.27. *With the same terminology,*

$$\frac{AL}{AG_e} = \frac{AG_e + LG_e}{AG_e} = 1 + \frac{LG_e}{AG_e} = 1 + \frac{(p-b)(p-c)}{a(p-a)}.$$

Lemma 8.28. *We have*

$$\mathbb{W} = \frac{r}{p-a}\left(\frac{a(p-a)}{(p-b)(p-c)} + 1\right).$$

Proof. Using the well known formulas $\Delta^2 = p(p-a)(p-b)(p-c)$ and $r = \Delta/p$, we get

$$\begin{aligned}
\frac{r}{p-a}\left(\frac{a(p-a)}{(p-b)(p-c)}+1\right) &= \frac{ar}{(p-b)(p-c)}+\frac{r}{p-a}\\
&= r\cdot\frac{a(p-a)+(p-b)(p-c)}{(p-a)(p-b)(p-c)}\\
&= \frac{\Delta}{p}\cdot\frac{a(p-a)+(p-b)(p-c)}{(p-a)(p-b)(p-c)}\\
&= \frac{a(p-a)+(p-b)(p-c)}{\Delta}\\
&= \frac{a\cdot\frac{b+c-a}{2}+\frac{(a-b+c)(a+b-c)}{4}}{\Delta}\\
&= \frac{2ab+2bc+2ac-a^2-b^2-c^2}{4\Delta}\\
&= \frac{4pr\mathbb{W}}{4\Delta} \qquad\text{(by Lemma 8.24)}\\
&= \mathbb{W}.
\end{aligned}$$

□

Theorem 8.29 (Radius of Inner Apollonius Circle)**.** *Let ρ_i be the radius of the inner Apollonius circle of γ_a, γ_b, and γ_c. Then*

$$\rho_i = rt\mathbb{W} - r.$$

Proof. By Corollary 8.17, C_i and the incircle are homothetic with G_e being the external center of similitude. Under this homothety, U_a maps to L. Thus

$$\frac{G_eU_a}{G_eL} = \frac{\rho_i}{r}.$$

We can write this as

$$\frac{\rho_i}{r} = \frac{AG_e - AU_a}{AL - AG_e} = \frac{AG_e - AL\cdot\frac{\rho_a}{r}}{AL - AG_e} = \frac{1-\frac{AL}{AG_e}\cdot\frac{\rho_a}{r}}{\frac{AL}{AG_e}-1}$$

because $AU_a = AL\cdot\frac{\rho_a}{r}$ (from Theorem 4.14). Now

$$\frac{AL}{AG_e} = \frac{AG_e + LG_e}{AG_e} = 1 + \frac{LG_e}{AG_e}.$$

From Lemma 8.26, we have

$$\frac{LG_e}{AG_e} = \frac{(p-b)(p-c)}{a(p-a)}$$

so

$$\frac{\rho_i}{r} = \frac{1-\left(1+\frac{(p-b)(p-c)}{a(p-a)}\right)\cdot\frac{\rho_a}{r}}{\frac{(p-b)(p-c)}{a(p-a)}}$$

which is equivalent to

$$\frac{\rho_i}{r}+1 = \left(1-\frac{\rho_a}{r}\right)\left(\frac{a(p-a)}{(p-b)(p-c)}+1\right).$$

We also know that

$$1-\frac{\rho_a}{r} = \frac{rt}{p-a}$$

from Corollary 4.7. Thus

$$\frac{\rho_i}{r}+1=\frac{rt}{p-a}\left(\frac{a(p-a)}{(p-b)(p-c)}+1\right).$$

By Lemma 8.28, this reduces to

$$\frac{\rho_i}{r}+1=t\mathbb{W},$$

so $\rho_i/r=t\mathbb{W}-1$ or $\rho_i=rt\mathbb{W}-r$. □

Note that r_i will be negative if the inner Apollonius circle is internally tangent to γ_a, γ_b, and γ_c. This will happen if $t\mathbb{W}<1$.

Open Question 3. *Is there a simpler proof of Theorem 8.29?*

Corollary 8.30. *We have*

$$\rho_i=2r-(\rho_a+\rho_b+\rho_c).$$

Proof. From Theorem 7.5, we have $rt\mathbb{W}=3r-(\rho_a+\rho_b+\rho_c)$. Therefore, we have $\rho_i=rt\mathbb{W}-r=3r-(\rho_a+\rho_b+\rho_c)-r=2r-(\rho_a+\rho_b+\rho_c)$. □

Theorem 8.31. *The circles γ_a, γ_b, and γ_c meet in a point (Figure 92) if and only if $t=1/\mathbb{W}$.*

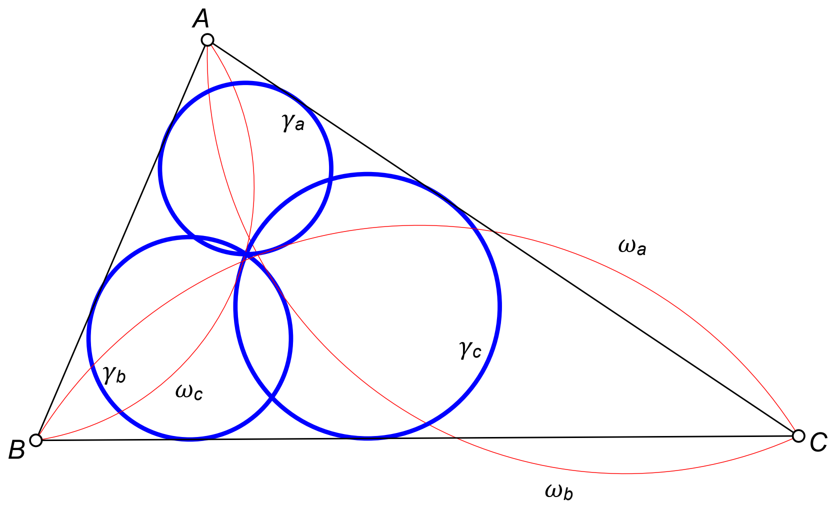


FIGURE 92. γ_a, γ_b, γ_c concur

Proof. The three circles concur if and only if the radius of the inner Apollonius circle is 0, that is, when $\rho_i=0$. By Theorem 8.29, $\rho_i=rt\mathbb{W}-r$. So $\rho_i=0$ if and only if $r=rt\mathbb{W}$ or $1=t\mathbb{W}$ since $r>0$. In other words, when $t=1/\mathbb{W}$. □

Corollary 8.32. *If $t=1/\mathbb{W}$, the circles γ_a, γ_b, and γ_c all pass through G_e, the Gergonne point of $\triangle ABC$.*

Proof. The common chord of each pair of circles is the radical axis of those two circles. Since the three common chords meet at the point of concurrence of the three circles, this point must be the radical center of the three circles. By Theorem 6.4, this is the Gergonne point of $\triangle ABC$. □

Theorem 8.33. *The circles ω_a, ω_b, and ω_c meet in a point (Figure 93) if and only if $\theta = 120°$.*

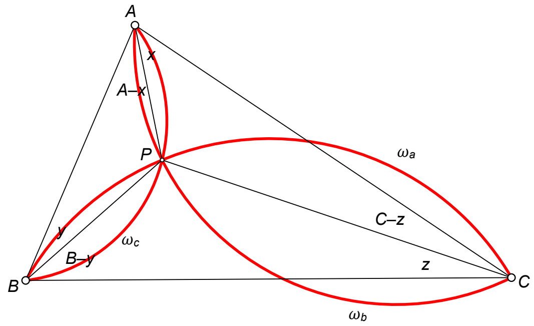


Figure 93. ω_a, ω_b, ω_c concur

Proof. Suppose the three arcs meet at P. Let $\angle PAC = x$, $\angle PBA = y$, and $\angle PCB = z$. Then $\angle BAP = A - x$, $\angle CBP = B - y$, and $\angle ACP = C - z$. An angle inscribed in a circle is measured by half its intercepted arc. So

$$2(A - x) + 2y = \theta,$$
$$2(B - y) + 2z = \theta,$$
$$2(C - z) + 2x = \theta.$$

Adding these three equations gives

$$3\theta = 2(A + B + C) = 2(180°) = 360°$$

or $\theta = 120°$. □

Theorem 8.34 (Radius of Outer Apollonius Circle)**.** *Let ρ_o be the radius of the outer Apollonius circle of γ_a, γ_b, and γ_c. Then*

$$\rho_o = \frac{r}{3}t\mathbb{W} + r.$$

Proof. Let G_e be the Gergonne point of $\triangle ABC$. Similar to Corollary 8.17, C_o and the incircle are homothetic with G_e being the external center of similitude. Let U_a and V_a be the points where AG_e meets γ_a, with V_a closer to A. Let W_a be the point where U_aV_a intersects the incircle (Figure 94).

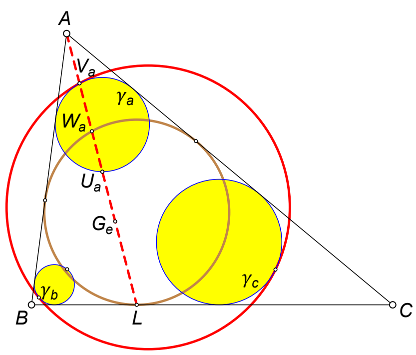


Figure 94. brown and red circles are homothetic at G_e

Under this homothety, V_a maps to W_a. Thus

$$\frac{\rho_o}{r} = \frac{G_eV_a}{G_eW_a} = \frac{AG_e - AV_a}{AG_e - AW_a}$$

$$(19) \qquad = \frac{AG_e/AL - AV_a/AL}{AG_e/AL - AW_a/AL},$$

From Theorem 4.17,

$$AV_a = \frac{(p-a-rt)\sqrt{a(p-a)}}{\sqrt{ap-(b-c)^2}}.$$

From Corollary 8.27,

$$\frac{AL}{AG_e} = 1 + \frac{(p-b)(p-c)}{a(p-a)}$$

so

$$\frac{AG_e}{AL} = \frac{a(p-a)}{a(p-a)+(p-b)(p-c)}.$$

From the homothety, center A that maps γ_a to the incircle,

$$\frac{AU_a}{AL} = \frac{\rho_a}{r}.$$

From Corollary 4.18, we have

$$\frac{AV_a}{AU_a} = \frac{a(p-a)}{ap-(b-c)^2}$$

Multiplying the previous two equations gives

$$\frac{AV_a}{AL} = \frac{a(p-a)}{ap-(b-c)^2} \cdot \frac{\rho_a}{r}.$$

From the homothety, center A that maps γ_a to the incircle,

$$\frac{AW_a}{AV_a} = \frac{r}{\rho_a}.$$

Multiplying the previous two equations gives

$$\frac{AW_a}{AL} = \frac{a(p-a)}{ap-(b-c)^2}.$$

Substituting the values for the ratios found into equation (19) gives

$$\frac{\rho_o}{r} = \frac{AG_e/AL - AV_a/AL}{AG_e/AL - AW_a/AL},$$

$$= \frac{\frac{a(p-a)}{a(p-a)+(p-b)(p-c)} - \frac{a(p-a)}{ap-(b-c)^2} \cdot \frac{\rho_a}{r}}{\frac{a(p-a)}{a(p-a)+(p-b)(p-c)} - \frac{a(p-a)}{ap-(b-c)^2}}.$$

Simplifying this algebraically gives

$$\frac{\rho_o}{r} = 1 + \frac{2rt}{3} \cdot \frac{2ab+2bc+2ca-a^2-b^2-c^2}{(a+b-c)(b+c-a)(c+a-b)}.$$

Applying Lemma 8.24 gives

$$\begin{aligned}\frac{\rho_o}{r}-1&=\frac{2rt}{3}\cdot\frac{4rp\mathbb{W}}{(a+b-c)(b+c-a)(c+a-b)}\\&=\frac{2rt}{3}\cdot\frac{4rp\mathbb{W}}{8(p-c)(p-a)(p-b)}\\&=\frac{t}{3}\cdot\frac{(rp)^2\mathbb{W}}{p(p-c)(p-a)(p-b)}\\&=\frac{t}{3}\cdot\frac{\Delta^2\mathbb{W}}{\Delta^2}\\&=\frac{t\mathbb{W}}{3},\end{aligned}$$

using the well-known formulas $\Delta = rp$ and $\Delta = \sqrt{p(p-a)(p-b)(p-c)}$. Thus, $\rho_o = rt\mathbb{W}/3 + r$. □

Open Question 4. *Is there a simpler proof of Theorem 8.34?*

Corollary 8.35. *We have*

$$\frac{\rho_i + r}{\rho_o - r} = 3.$$

Proof. This follows immediately from Theorems 8.29 and 8.34. □

Remember when applying this result, that ρ_i is to be considered negative when the inner Apollonius circle is internally tangent to γ_a, γ_b, and γ_c, as shown in Figure 96.

The line through the incenter of a triangle and the Gergonne point of that triangle is called the *Soddy line* of the triangle.

Theorem 8.36. *For a general triad of circles associated with $\triangle ABC$, let U be the center of the inner Apollonius circle of γ_a, γ_b, and γ_c. Let V be the center of the outer Apollonius circle of γ_a, γ_b, and γ_c. Then U and V lie on the Soddy line of the triangle and $UI : IV = 3 : 1$ (Figure 95).*

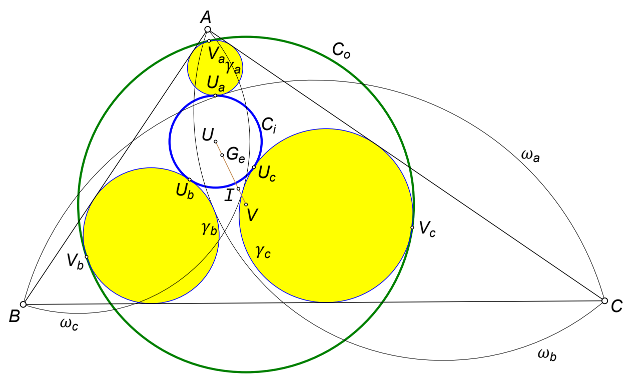


Figure 95. $UI : IV = 3 : 1$

Proof. All distances along the Soddy line will be signed. We have

$$\frac{UI}{G_eI} = \frac{UG_e + G_eI}{G_eI} = \frac{UG_e}{G_eI} + 1 = \frac{\rho_i}{r} + 1 = \frac{\rho_i + r}{r}$$

and
$$\frac{IV}{G_eI} = \frac{G_eV - G_eI}{G_eI} = \frac{G_eV}{G_eI} - 1 = \frac{\rho_o}{r} - 1 = \frac{\rho_o - r}{r}.$$
Dividing gives
$$\frac{UI}{IV} = \frac{\rho_i + r}{\rho_o - r}.$$
The result now follows from Corollary 8.35. □

Open Question 5. *Is there a simple geometric proof that $UI/IV = 3$?*

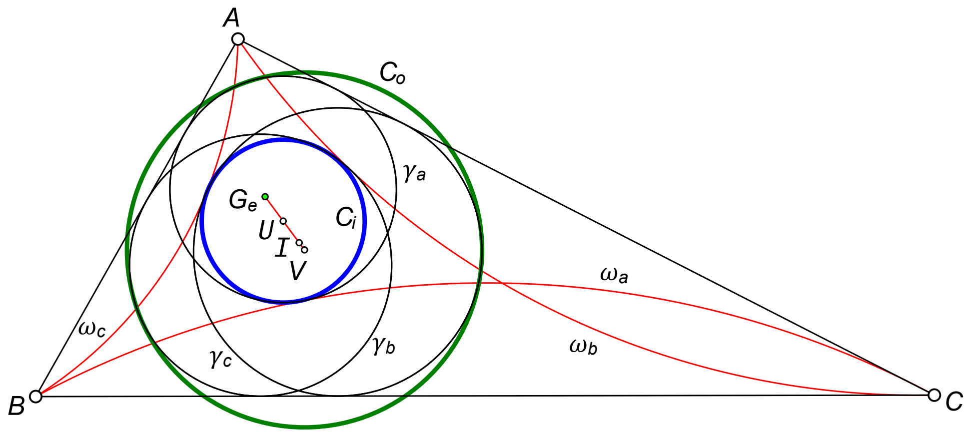


FIGURE 96. $UI : IV = 3 : 1$

If the inner Apollonius circle is internally tangent to the three circles as in Figure 96, then U and V still lie on the Soddy line, but the points on that line occur in the order G_e, U, I, V.

Corollary 8.37. *We have*
$$\frac{G_eI}{IV} = \frac{r}{\rho_o - r}.$$

Corollary 8.38. *We have the extended proportion*
$$UG_e : G_eI : IV = \rho_i : r : \rho_o - r.$$

It should be noted that the distance from G_e to I in terms of parts of the triangle is known. From [27, p. 184], we have the following result.

Theorem 8.39. *We have*
$$G_eI = \frac{r}{4R + r}\sqrt{(4R + r)^2 - 3p^2} = r\sqrt{1 - 3/\mathbb{W}^2}.$$

This allows us to express any of the distances between the points U, V, G_e, and I in terms of R, r, and p by using Corollary 8.38.

Theorem 8.40. *For a general triad of circles associated with $\triangle ABC$, the inradius r and the radii ρ_i and ρ_o of the Apollonius circles satisfy the relation*
$$3\rho_o = \rho_i + 4r.$$

Proof. This is algebraically equivalent to Corollary 8.35. □

Theorem 8.41. *For a general triad of circles associated with $\triangle ABC$, the radii ρ_i and ρ_o of the Apollonius circles and the radii ρ_a, ρ_b, and ρ_c of the three circles in the triad satisfy the relation*

$$3\rho_o = 2(\rho_a + \rho_b + \rho_c) + 3\rho_i.$$

Proof. From Theorem 7.5, we have

$$\rho_a + \rho_b + \rho_c = 3r - rt\mathbb{W}.$$

From Theorems 8.29 and 8.34, we have

$$3\rho_o - 3\rho_i = 6r - 2rt\mathbb{W} = 2(\rho_a + \rho_b + \rho_c)$$

as desired. ☐

Theorem 8.42. *Let ρ_i be the radius of the inner Apollonius circle of γ_a, γ_b, and γ_c. Then*

$$\rho_i^2 = \rho_a^2 + \rho_b^2 + \rho_c^2 + 2r^2\left(t^2 - 1\right).$$

Proof. This follows algebraically by combining Theorems 8.29 and 7.8. ☐

When $\theta = 180°$, the arcs become semicircles, $t = 1$, and this result agrees with Theorem 6.1 in [30].

9. Relationship with Semicircles

Many of the elements of our configuration are proportional to the corresponding elements when ω_a, ω_b, and ω_c are semicircles (i.e. when $\theta = 180°$ or $t = 1$).

If x is any measurement or object, let x^* denote the same measurement or object when $\theta = 180°$, i.e. when the arcs are semicircles.

In [30], it was found that

$$\rho_a^* = r\left(1 - \tan\frac{A}{2}\right)$$
$$\rho_b^* = r\left(1 - \tan\frac{B}{2}\right)$$
$$\rho_c^* = r\left(1 - \tan\frac{C}{2}\right)$$

and in Theorem 4.6, we found that

$$\rho_a = r\left(1 - \tan\frac{A}{2}\tan\frac{\theta}{4}\right)$$
$$\rho_b = r\left(1 - \tan\frac{B}{2}\tan\frac{\theta}{4}\right)$$
$$\rho_c = r\left(1 - \tan\frac{C}{2}\tan\frac{\theta}{4}\right)$$

In other words, if $t = \tan\frac{\theta}{4}$, then we have the following results.

Theorem 9.1. *The following identities are true.*

$$\rho_a - r = t(\rho_a^* - r)$$
$$\rho_b - r = t(\rho_b^* - r)$$
$$\rho_c - r = t(\rho_c^* - r)$$

Theorem 9.2. *The following identities are true.*

$$\rho_a - \rho_b = t(\rho_a^* - \rho_b^*)$$
$$\rho_b - \rho_c = t(\rho_b^* - \rho_c^*)$$
$$\rho_c - \rho_a = t(\rho_c^* - \rho_a^*)$$

Let T_{bc} denote the length of the common external tangent between circles γ_b and γ_c. Define T_{ab} and T_{ca} similarly.

In [30], it was found that $T_{ab}^* = T_{bc}^* = T_{ca}^* = 2r$. In Theorem 6.1, we found that $T_{ab} = T_{bc} = T_{ca} = 2rt$. This gives us the following result.

Theorem 9.3. *The following identities are true.*

$$T_{ab} = tT_{ab}^*$$
$$T_{bc} = tT_{bc}^*$$
$$T_{ca} = tT_{ca}^*$$

Using these, we can prove the following new result.

Theorem 9.4. *Let D_{bc} denote the distance between the centers of γ_b and γ_c. Define D_{ab} and D_{ca} similarly. Then the following identities are true.*

$$D_{ab} = tD_{ab}^*$$
$$D_{bc} = tD_{bc}^*$$
$$D_{ca} = tD_{ca}^*$$

Proof. By symmetry, it suffices to prove the result for D_{bc}. Let E be the center of γ_b and let F be the center of γ_c. Let the common external tangent along BC be XY as shown in Figure 97. Let the foot of the perpendicular from E to FY be H. Then in right triangle EHF, we have $EH = XY = T_{bc} = tT_{bc}^*$ by Theorem 9.3. We also have $FH = |\rho_c - \rho_b|$. By Theorem 9.2, $FH = t|\rho_c^* - \rho_a^*|$. Since $\triangle EHF \sim \triangle E^*H^*F^*$, we must therefore have $D_{bc} = tD_{bc}^*$. □

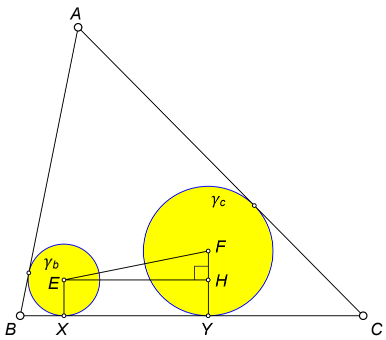


FIGURE 97. case where H lies between F and Y

Corollary 9.5. *The triangles formed by the centers of γ_a, γ_b, γ_c and γ_a^*, γ_b^*, γ_c^* are similar.*

Theorem 9.6. *We have*

$$\rho_i + r = t(\rho_i^* + r).$$

Proof. This follows immediately from Theorem 8.29. □

Theorem 9.7. *We have*
$$\rho_o - r = t(\rho_o^* - r).$$

Proof. This follows immediately from Theorem 8.34. □

Theorem 9.8. *Let γ_a, γ_b, and γ_c be a general triad of circles associated with $\triangle ABC$. Let U be the center of the inner Apollonius circle of the circles in the triad. Let d_a denote the distance between U and the center of γ_a. Define d_b and d_c similarly. Then the following identities are true.*
$$d_a - d_b = t(d_a^* - d_b^*)$$
$$d_b - d_c = t(d_b^* - d_b^*)$$
$$d_c - d_a = t(d_c^* - d_a^*)$$

Proof. Let ρ_i be the radius of the inner Apollonius circle. Then $d_a = \rho_i + \rho_a$ (Figure 98).

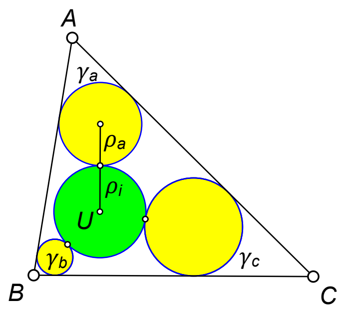


Figure 98. $d_a = \rho_i + \rho_a$

Similarly, $d_b = \rho_i + \rho_b$. Thus $d_a - d_b = \rho_a - \rho_b$. By Theorem 9.2,
$$d_a - d_b = t(\rho_a^* - \rho_b^*) = t(d_a^* - d_b^*).$$
The same argument works for $d_b - d_c$ and $d_c - d_a$. □

Theorem 9.9. *If $u = EF$, $v = DF$, $w = DE$ are the distances between the centers of the circles γ_a, γ_b, and γ_c, we have*
$$u^2 = \frac{a(p-a)[ap-(b-c)^2]t^2}{p^2},$$
$$v^2 = \frac{b(p-b)[bp-(c-a)^2]t^2}{p^2},$$
$$w^2 = \frac{c(p-c)[ap-(a-b)^2]t^2}{p^2}.$$

Proof. This follows from Theorem 9.4 and the simplified values of u^*, v^*, and w^* found in [30]. □

10. Variations

We have studied the case where Ajima circle γ_a is inscribed in $\angle BAC$ and is inside $\triangle ABC$ and is externally tangent to circle ω_a.

There are actually four circles that are inscribed in $\angle BAC$ and are tangent to circle ω_a. These circles are shown in Figure 99.

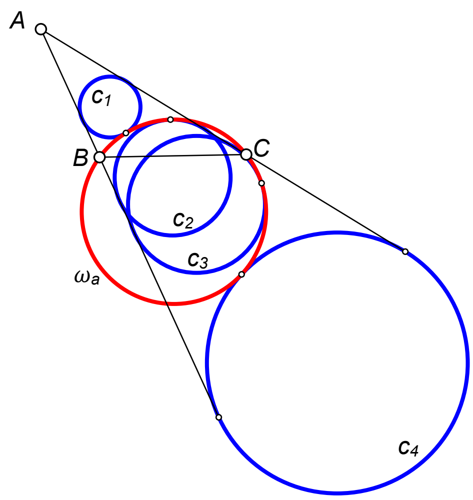


FIGURE 99. circles inscribed in $\angle A$ and tangent to red circle

Circle c_1 is variation 1 and is the variation already studied. Note that circle c_4 is inscribed in $\angle BAC$ and is *outside* $\triangle ABC$ as well as being externally tangent to circle ω_a. Circle c_2 and c_3 are *internally* tangent to ω_a. The touch point of c_2 and ω_a is inside $\triangle ABC$ while the touch point of c_3 and ω_a is outside $\triangle ABC$.

Many of the results we found for variation 1 work for the other variations as well. We present below a few of these results. Proofs are omitted because they are similar to the proofs given for variation 1. Variants 2, 3, and 4, are shown in Figure 100.

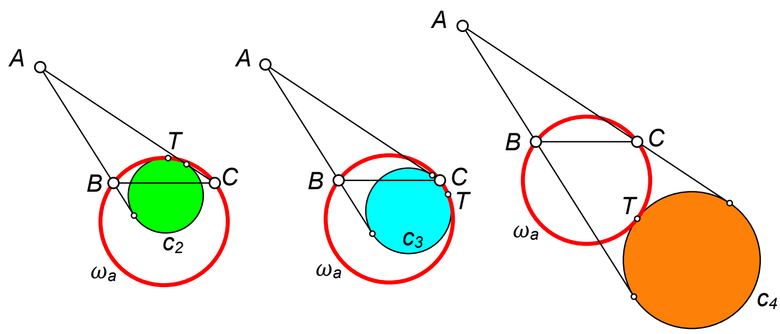


FIGURE 100. variants 2 (green), 3 (blue), and 4 (orange)

The Catalytic Lemma

The Catalytic Lemma remains true except in some cases where the incenter is replaced by an excenter. Figure 101 shows variants 2, 3, and 4. In each case, B, K, T, and an incenter/excenter lie on a circle.

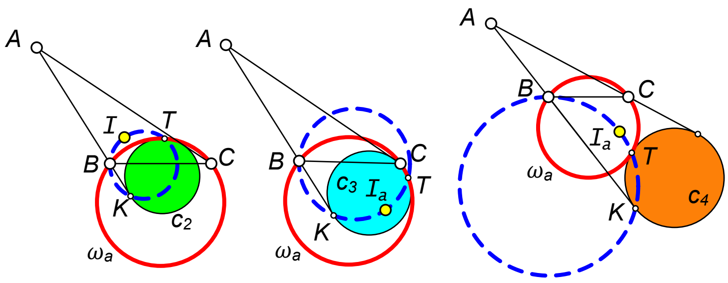


FIGURE 101. four points lie on a circle

Protasov's Theorem

Protasov's Theorem remains true except in some cases where the incenter is replaced by an excenter. Figure 102 shows variants 2, 3, and 4. In each case, the blue line bisects the angle formed by the dashed lines.

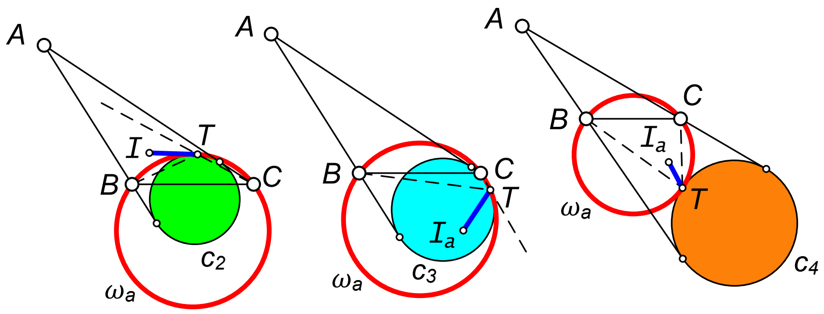


FIGURE 102. blue line bisects angle formed by dashed lines

Theorem 2.10

Theorem 2.10 remains true except in some cases where the incenter is replaced by an excenter. Figure 103 shows variants 2, 3, and 4. In each case, the parallel to BE through an incenter or excenter meets AC at F, where E is the point where ω_a meets AC. Then $IF = FK$ where K is the point where γ_a touches AC.

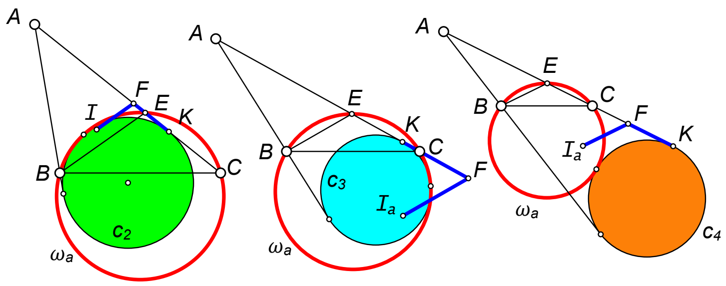


FIGURE 103. blue lines are congruent

Theorem 2.7

Theorem 2.7 remains true except in some cases where the incenter is replaced by an excenter. Figure 103 shows variants 2, 3, and 4. In each case, the line through T and an incenter or excenter meets ω_a at a point on the perpendicular bisector of BC (opposite T).

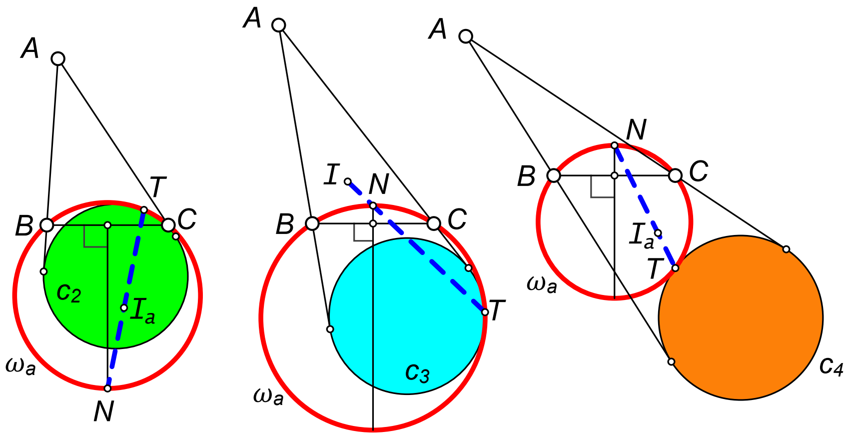


FIGURE 104. T, N, and an incenter or excenter lie on a line.

Ajima's Theorem

Similar formulas for the radii of the variant circles can be found similar to Ajima's Theorem. These are shown in Figure 105, where r_a denotes the radius of the A-excircle of $\triangle ABC$.

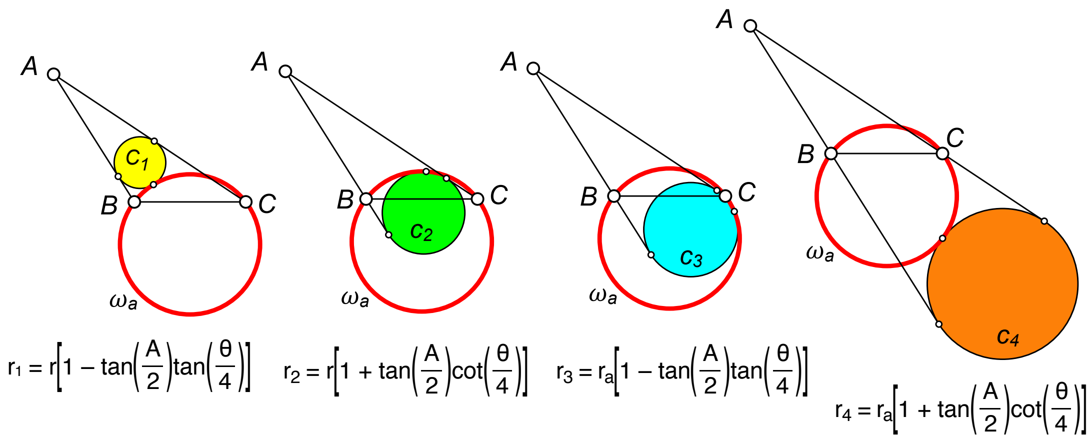


FIGURE 105.

The Paasche Analog

The Paasche Analog (Theorem 6.7) remains true. If γ_a is any one of these variant circles, and if T_a is the touch point of γ_a with ω_a, with T_b and T_c defined similarly, then AT_a, BT_b, and CT_c are concurrent. See Figure 106 for one case.

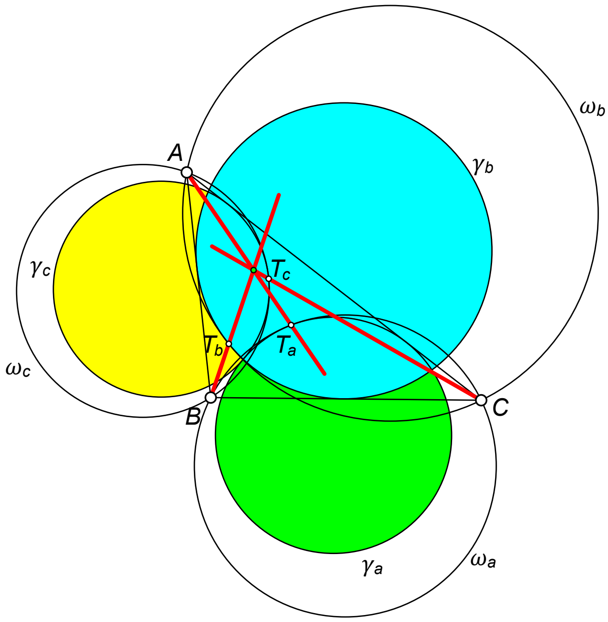


Figure 106.

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