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Notes on the Six-Segment Theorem

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Abstract. The six-segment theorem (*rokushajutsu* 六斜術) seems to have been discussed previously only in books and articles in Japanese. I present it here in the form of three proofs of the theorem and two illustrative examples of its use, with references to relevant Japanese sources.

Keywords. *wasan*, trigonometry, linear algebra.

Mathematics Subject Classification (2020). 51M15, 51M25

1. INTRODUCTION

The six-segment theorem states that, for the lengths shown in Figure 1,

$$\begin{aligned}(a^2+y^2-z^2)^2x^2+(b^2+z^2-x^2)^2y^2+(c^2+x^2-y^2)^2z^2&\\ =4x^2y^2z^2+(a^2+y^2-z^2)(b^2+z^2-x^2)(c^2+x^2-y^2)&\end{aligned} \quad (1).$$

Here, a, b, c are the side lengths of triangle ABC and x, y, z are the distances of A, B, C, respectively, from some fourth point P in the plane. There are at least three ways to prove this theorem, and it turns out to be unusually powerful.

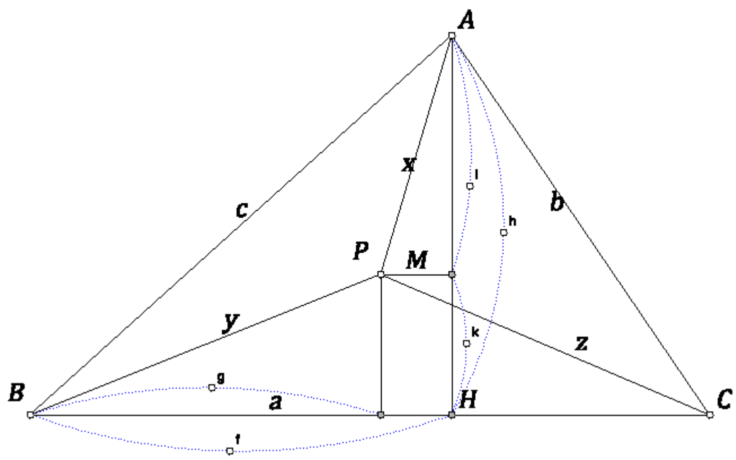

Figure 1. The basic figure.

2. TRIGONOMETRIC PROOF

For a quick proof ([9], [10]), let the central angles $\angle CPA, \angle BPC, \angle APB$, respectively, be α, β, γ, and note that $\cos\gamma = \cos(\alpha+\beta)$. Hence

$$\begin{aligned}\cos^2\alpha + \cos^2\beta + \cos^2\gamma &= \frac{1+\cos 2\alpha}{2} + \frac{1+\cos 2\beta}{2} + \cos^2\gamma \\ &= 1 + \frac{\cos 2\alpha + \cos 2\beta}{2} + \cos^2\gamma \\ &= 1 + \cos(\alpha+\beta)\cos(\alpha-\beta) + \cos^2\gamma \\ &= 1 + \cos\gamma\cos(\alpha-\beta) + \cos\gamma\cos(\alpha+\beta) \\ &= 1 + \cos\gamma\,[\cos(\alpha-\beta) + \cos(\alpha+\beta)] \\ &= 1 + \cos\gamma\,[2\cos\alpha\cos\beta] \\ &= 1 + 2\cos\alpha\cos\beta\cos\gamma.\end{aligned}$$

Applying the Law of Cosines to this equation term by term and simplifying,

$$\begin{aligned}&\left(\frac{y^2+z^2-a^2}{2yz}\right)^2 + \left(\frac{z^2+x^2-b^2}{2zx}\right)^2 + \left(\frac{x^2+y^2-c^2}{2xy}\right)^2 \\ &= 1 + 2\left(\frac{y^2+z^2-a^2}{2yz}\right)\left(\frac{z^2+x^2-b^2}{2zx}\right)\left(\frac{x^2+y^2-c^2}{2xy}\right) \\ &= 1 + \frac{(y^2+z^2-a^2)(z^2+x^2-b^2)(x^2+y^2-c^2)}{4x^2y^2z^2},\end{aligned}$$

which becomes (1) when we multiply it through by $4x^2y^2z^2$.

3. GEOMETRIC PROOF WITHOUT ANGLES

The *wasanka* eschewed trigonometry, so they would have derived (1) differently. According to Kotera (2009), they would have defined

$$f = \frac{(c^2+a^2-b^2)}{2a},\ g = \frac{(y^2+a^2-z^2)}{2a},$$

$$h = \sqrt{c^2 - f^2},\ i = \sqrt{y^2 - g^2},\ k = \sqrt{x^2 - (f-g)^2}$$

(see again Figure 1), and then solved $h - i = k$, i.e.

$$\sqrt{c^2 - f^2} - \sqrt{y^2 - g^2} = \sqrt{x^2 - (f-g)^2},$$

by squaring, rearranging, squaring again, replacing f and g, and simplifying. This produces

$$\begin{aligned} a^2x^2(b^2+c^2+y^2+z^2-a^2-x^2) &+ b^2y^2(c^2+a^2+z^2+x^2-b^2-y^2) \\ &+ c^2z^2(a^2+b^2+x^2+y^2-c^2-z^2) \\ &= a^2b^2c^2 + a^2y^2z^2 + b^2z^2x^2 + c^2x^2y^2, \end{aligned} \tag{2}$$

which is equivalent to (1). For if we move all the terms in (1) to the left side of the equation, expand, and segregate positive and negative terms, we obtain

$$\begin{aligned} a^2b^2c^2 + a^2y^2z^2 + b^2z^2x^2 + c^2y^2x^2 &+ a^4x^2 + a^2x^4 + b^4y^2 + b^2y^4 + c^4z^2 + c^2z^4 \\ &- (a^2b^2x^2 + a^2c^2x^2 + a^2x^2y^2 + a^2x^2z^2 + b^2c^2y^2 + a^2b^2y^2 + b^2y^2z^2 \\ &+ b^2x^2y^2 + a^2c^2z^2 + b^2c^2z^2 + c^2x^2z^2 + c^2y^2z^2) = 0. \end{aligned}$$

Now factor the left side to eliminate fourth powers:

$$\begin{aligned} a^2b^2c^2 + a^2y^2z^2 + b^2z^2x^2 + c^2y^2x^2 &+ [a^2x^2(a^2+x^2) + b^2y^2(b^2+y^2) + c^2z^2(c^2+z^2)] \\ &- [a^2x^2(b^2+c^2+y^2+z^2) + b^2y^2(c^2+a^2+z^2+x^2) \\ &+ c^2z^2(a^2+b^2+x^2+y^2)] = 0. \end{aligned}$$

With a bit more factoring, this becomes

$$\begin{aligned} a^2b^2c^2 + a^2y^2z^2 + b^2z^2x^2 + c^2x^2y^2 &- [a^2x^2(b^2+c^2+y^2+z^2-a^2-x^2) \\ &+ b^2y^2(c^2+a^2+z^2+x^2-b^2-y^2) + c^2z^2(a^2+b^2+x^2+y^2-c^2-z^2)] \\ &= 0, \end{aligned}$$

which is clearly a rearrangement of (2).

4. A THOROUGHLY MODERN PROOF

Hirata ([4]) offers an elegant proof using equation (3), which is the six-segment theorem for the quadrilateral $ABCO$ and its diagonals as labeled in Figure 2:

$$a^2x^2(-a^2+b^2+c^2-x^2+y^2+z^2)+b^2y^2(a^2-b^2+c^2+x^2-y^2+z^2)+ \\ c^2z^2(a^2+b^2-c^2+x^2+y^2-z^2)-(a^2b^2z^2+b^2c^2x^2+c^2a^2y^2+x^2y^2z^2)=0 \quad (3).$$

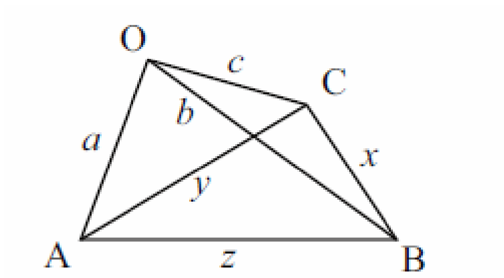

Figure 2. Hirata's relabelling.

Let $H(x,y,z)=(x+y+z)(-x+y+z)(x-y+z)(x+y-z)$, where x,y,z are the lengths of the sides of triangle ABC. Expanding $H(x,y,z)$, we get $2(x^2y^2+y^2z^2+z^2x^2)-x^4-y^4-z^4$, in which only even powers of x,y,z appear; therefore, for all real x,y,z, we have $H(x,y,z)\geq 0$. Furthermore, writing s for the semiperimeter $\frac{x+y+z}{2}$, we have $H(x,y,z)=16s(s-x)(s-y)(s-z)$; in light of Heron's formula for the area S of ABC, this means that $H(x,y,z)=16S^2$. Therefore, in Figure 2, A,B,C are collinear if and only $H(x,y,z)=0$.

The foregoing is a special case of a more general theorem of linear algebra: *the Gram determinant of a set of vectors is zero if and only if the vectors are linearly dependent.* In the case of Figure 2, we have just three vectors $\mathbf{a}=\overrightarrow{OA},\mathbf{b}=\overrightarrow{OB},\mathbf{c}=\overrightarrow{OC}$. Their Gram determinant is $G(\mathbf{a},\mathbf{b},\mathbf{c})=\begin{vmatrix}\mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c}\\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c}\\ \mathbf{c}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{c}\end{vmatrix}$, or

$$\begin{vmatrix} a^2 & \frac{a^2+b^2-z^2}{2} & \frac{a^2+c^2-y^2}{2} \\ \frac{a^2+b^2-z^2}{2} & b^2 & \frac{b^2+c^2-x^2}{2} \\ \frac{a^2+c^2-y^2}{2} & \frac{b^2+c^2-x^2}{2} & c^2 \end{vmatrix}, \quad (4)$$

where we have made use of the commutativity of the inner product and the Law of Cosines repeatedly to rewrite elements of the determinant in terms of scalars. Because **a**, **b**, **c** are coplanar, they are linearly dependent (any one of them can be expressed as a linear combination of the other two), so (4), i.e. $G(\mathbf{a}, \mathbf{b}, \mathbf{c})$, must be zero. Since the expansion of (4) is just 4 times the left side of (3), this verifies the *rokushajutsu* equation.[2]

5. A SIMPLE APPLICATION

Fukagawa and Rothman ([3]: 262) quote the following problem from the travel diary of the *wasanka* Yamaguchi Kazu [penname (*gō*) Kanzan] 山口和[坎山] (1781–1850), but offer no solution. It is, however, included in the famous book *Sanpō shōjo*, and the following solution is based largely on Kotera Hiroshi's commentary on it [5]. Kotera notes that Aida Yasuaki 会田安明 (1747–1817) gives this problem in fascicle 49 of *Sanpō tensei hō*, without solution, ascribing it to Itō Shūin 伊藤秀允 (1755–1838).[3]

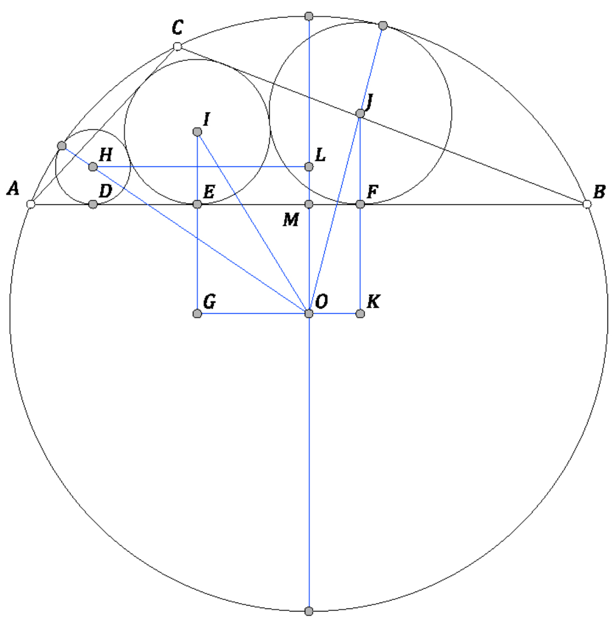

Figure 3. Itō's problem.

Given $(H)u, (I)r, (J)v,$ and $(O)R$ as shown in Figure 3, express R in terms of u, v, r.

Consider triangle HIL and the fourth point O. We see that $HI = u + r$ and $IL = v + r$. Letting $d = HL + OK$, we have $HJ^2 = d^2 + (u - v)^2$. Substitute $HI, IL,$ and HJ^2 for $c, b,$ and a^2 in (1)

[2] See also [13, 68–69].

[3] Kotera's version of (1) accidentally omits the exponents of the quantities $(a^2 + y^2 - z^2)$, $(b^2 + z^2 - x^2)$, and $(c^2 + x^2 - y^2)$ on the left side of (1).

or (2). Clearly, $y = OH = R - u$, $z = OJ = R - v$, and (thanks to Euler) $x^2 = OI^2 = R(R - 2r)$. Simplifying coefficients, this yields

$$R^2[d^4 + 16r^2(u - v)^2 - 8d^2r(u + v)] + 2rR\{4r^2(u - v)^2 + d^2[2r^2 + 8uv + 3r(u + v)] - d^4\} + r^2[r^2(u - v)^2 + d^2(r + 2u)(r + 2v)] = 0.$$

Because of a well-known *wasan* theorem about the external bitangent segment joining externally tangent circles, we can substitute $2\sqrt{ru} + 2\sqrt{rv}$ for d. When we do, the coefficient of R^2 vanishes, the coefficient of R becomes $r^3(\sqrt{u} + \sqrt{v})^2(4r^2 + 9ru - 2r\sqrt{u}\sqrt{v} + 9rv + 16uv)$, and the constant term becomes $r^3\left(\sqrt{u} + \sqrt{v}\right)^2\left(4r^2 + 9ru - 2\sqrt{u}\sqrt{v} + 9rv + 16uv\right)$. From this, it follows quickly that

$$R = \frac{r(4r^2 + 9ru - 2r\sqrt{u}\sqrt{v} + 9rv + 16uv)}{16(r - 4\sqrt{u}\sqrt{v})(r - \sqrt{u}\sqrt{v})}.$$

Even if R, r, u, v are taken to be diameters, as was usual in *wasan* problems, rather than radii, this is still the formally correct solution.

It is worth adding that constructing (H) and (J) in Figure 3 exactly is not a trivial matter. A modern way to do so (easily carried out using a program such as Cinderella) is to use inversion in an arbitrary circle centered in I. Draw a dilated circle (O) with radius $R + r$ and a parallel to AB more remote from C by r. The inverses of this dilated circle and displaced line are circles. Construct their external bitangents and invert again to obtain two new circles. Their centers will be H and J, and reducing their radii by r produces circles that each touch (O) internally, (I) externally, and AB.

6. A CELEBRRATED PROBLEM

The Descartes Circle Theorem (DCT) is so called because Descartes presented it in a letter of 1634 (without a proof) to Elizabeth, Queen of Bohemia, then in exile in The Hague.[4] It may have been known by Apollonius of Perga (3rd c. BCE), but his book *On Tangencies* is lost. It was rediscovered in the 19th century by Steiner and by Beecroft, but Frederick Soddy, who had won the Nobel Prize for Chemistry in 1921, is usually credited with the first proof (1936), extending the theorem to three dimensions.[5] Nevertheless, Japanese of the 18th and 19th centuries knew and used this theorem.

Michiwaki and Kimura ([8] 161) cite *Sanpō ruiju, kan-san zeishiki endan* 『算法類聚』巻三贅式演段 (1751) by Yamaji Nushizumi 山路主住 (1724–1770) as the earliest Japanese mention. Aida Yasuaki ([1] 3:27–28) gave a proof of the DCT in 1810, and Fukagawa and Rothman ([3]:

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[4] Elizabeth Stuart, daughter of James I of England, married Frederick V, Elector Palatine of the Rhine, whose coronation as King of Bohemia in 1619 sparked the Thirty Years' War; their grandson became George I of the United Kingdom in 1714.

[55] On extension to higher dimensions, see [6].

289–91) give a proof based on lemma 51 in the Edo-period work *Sanpō jojutsu* of 1841 [12]. (They also explain Soddy's generalization.) In fact, the DCT is lemma 55 in the same work. but, for our purposes, the interesting point is that the DCT can be proven directly from the six-segment theorem, as noted in [7].

The DCT states that, if r_1, r_2, r_3 are the radii of three circles each of which touch the other two externally, and r is the radius of a fourth circle that touches all three, then $2\left(\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}+\frac{1}{r^2}\right)=\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\pm\frac{1}{r}\right)^2$, where the + sign applies when tangencies with the fourth circle are external and the – sign when they are internal.

Suppose the centers of the three-circle chain are vertices of triangle ABC. Then there is a numbering of r_i such that $a = r_1 + r_2$, $b = r_2 + r_3$, and $c = r_3 + r_1$, and $x = r \pm r_3, y = r \pm r_1, z = r \pm r_2$. For the sake of illustration, consider just the + case. Making substitutions in (1) and simplifying, we get

$$r_1^2 r_2^2 r_3^2 + r^2[r_1^2(r_2 - r_3)^2 + r_2^2 r_3^2 - 2r_1 r_2 r_3(r_2 + r_3)] = 2r r_2 r_2 r_3 [r_2 r_3 + r(r_2 + r_3)].$$

This is equivalent to the expansion of $2\left(\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}+\frac{1}{r^2}\right)=\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r}\right)^2$ multiplied through by $r r_1 r_2 r_3$. The proof for $2\left(\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}+\frac{1}{r^2}\right)=\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}-\frac{1}{r}\right)^2$ is analogous.[6]

Other examples of how *wasanka* made use of the six-segment theorem are discussed in [2] and [11].

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